Video Lecture G2: Orthogonal Sets & Spaces in $\mathbb{R}^n$

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Outline & Objectives

- Define the concept of an *orthogonal set* of vectors in $\mathbb{R}^n$ and prove that any such set is linearly independent (if all vectors are nonzero).
- Analyze the properties of *orthogonal bases* for subspaces W of $\mathbb{R}^n$, particularly how orthogonality simplifies writing any $w \in W$ in terms of the basis.
- Define the *orthogonal complement* $W^\perp$ of a subspace W of $\mathbb{R}^n$, and prove the fundamental theorem that $(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$ and $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$.

Orthogonal Sets

Definition (Orthogonal sets and bases)

Call a set of vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ an orthogonal set if $\vec{u}_i \cdot \vec{u}_j = 0 \; \forall \, i \neq j$. If additionally S is a basis for a subspace W, call S an orthogonal basis for W.

$\vec{u}_1 = (-2, 1, 3, 1)$, $\vec{u}_2 = (1, 0, 1, -1)$, $\vec{u}_3 = (1, 1, 0, 1)$ in $\mathbb{R}^4$.

Theorem (Orthogonal sets are linearly independent)

An orthogonal set of nonzero vectors $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is linearly independent, hence a basis for $W = \operatorname{Span} S$.

Proof: Apply $\cdot\vec{u}_j$ to a linear dependence relation for S.

Theorem (Orthogonal bases rock!)

Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ be an orthogonal basis for W and suppose $\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$. Then $c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} \; \forall \, j \in [p]$.

Orthogonal Complements

Definition (Orthogonal complement of a subspace W)

Call $\vec{z} \in \mathbb{R}^n$ orthogonal to W if $\vec{z} \cdot \vec{w} = 0 \ \forall \, \vec{w} \in W$. The orthogonal complement of W is $W^\perp := \{\vec{z} : \vec{z} \text{ is orthogonal to } W\}$

$$W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \implies W^\perp = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\mathrm{Span}\,\{(-2,1,3,1),(1,0,1,-1)\}^\perp = \mathrm{Span}\,\{(-1,-5,1,0),(1,1,0,1)\}.$$

Easy Ex: $W^\perp$ is a subspace and $W \cap W^\perp = \{\vec{0}\}$.

Theorem ("Fundamental Theorem" for matrix subspaces)

For any $A \in \mathbb{R}^{m \times n}$, $(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$ and $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$

Pf: By row/col rule for mx mult, each row of A is $\perp$ to any $x \in \mathrm{Nul}\, A$.

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 5 \\ 1 & -1 & -1 \end{bmatrix} \overset{\mathcal{R}}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad A^T = \begin{bmatrix} -2 & 1 & 3 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 5 & -1 \end{bmatrix} \overset{\mathcal{R}}{\sim} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$