Video Lecture F8: Diagonalization of \( \mathbb{R} \) Symmetric Matrices

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Outline & Objectives

- Combine the
  - geometric work we’ve done on orthogonality and projections with the
  - algebraic work on eigenvalues and change-of-basis

to analyze the following fundamental result:
Every symmetric matrix $A \in \mathbb{R}^{n \times n}$ is orthogonally diagonalizable: $A = PDP^{-1}$, where $P$ is an orthogonal matrix.

- Give an example of how to compute this in practice.
Symmetric Matrices & orthogonal diagonalization

Which are symmetric \((A = A^T)\)? If not, make them so!

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\]

Which matrices are orthogonal \((U^T = U^{-1})\)? If not...

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad P = \frac{1}{5} \begin{bmatrix} 3 & 4 & 3 \\ 3 & 4 & 3 \\ 0 & 0 & 4 \end{bmatrix}.
\]

Recall: A matrix \(A \in \mathbb{R}^{n \times n}\) is diagonalizable (i.e., similar to a diagonal matrix) \(\iff\) \(A\) has \(n\) lin indep eigenvectors, \(\{\vec{v}_i\}_{i=1}^n\). Then \(A\) factors as \(A = PD P^{-1}\), where \(D\) is diagonal and \(P\) is invertible. In fact, \(P = [\vec{v}_1 \vec{v}_2 \ldots \vec{v}_n]\), and \(D = \text{diag}(\lambda_1, \ldots, \lambda_n)\).

**Definition**

Call a matrix \(A \in \mathbb{R}^{n \times n}\) **orthogonally diagonalizable** if \(A = PD P^{-1}\) for some orthogonal matrix \(P\). (So \(A = PDP^T\) also.)
Spectral Theorem for Real Symmetric Matrices

**Proposition**

$A \in \mathbb{R}^{n \times n}$ is orthogonally diagonalizable $\implies A$ is symmetric.

**Pf:** $A = PDP^T \implies A^T = (PDP^T)^T = P^TP^TDP^T = PDP^T = A.$

**Lemma**

If $A$ is a (real) symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.

**Proof:** Let $\vec{v}$ correspond to $\lambda$ and $\vec{w}$ to $\mu$ with $\lambda \neq \mu$. Want to show that $\lambda(v \cdot w) = \mu(v \cdot w)$, forcing $v \cdot w = 0 \ldots \blacksquare$

**Theorem (Spectral Theorem for $\mathbb{R}$ symmetric matrices)**

Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then

1. $A$ has $n$ real eigenvalues (counting multiplicities);
2. $\dim E_\lambda = \text{mult} \lambda$ (as root of $\chi(\lambda)$);
3. For $\lambda \neq \mu$ in Spec $A$, $E_\lambda \perp E_\mu$;
4. $A$ is orthogonally diagonalizable.
An Example of Orthogonal Diagonalization

\[ A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} \]

Q: What’s an obvious eigenvalue/vector pair?

\((1, 1, 1)\) is an eigenvector with eigenvalue 6.

\[ \chi(\lambda) = \lambda^3 - 12\lambda^2 + 45\lambda - 54 = (\lambda - 6)(\lambda - 3)^2. \]

\[ A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \]

Q: Are the columns of \(P\) orthogonal? NO! What should we do?

Apply Gram-Schmidt to the columns of \(P\) as needed. Now divide each column through as appropriate to normalize. Get

\[ A = QDQ^{-1} = QDQ^T, \text{ where } Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \]

CHECK: \(Q\) is an orthogonal matrix.