Video Lecture F7: QR Factorization of Matrices

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Outline & Objectives

- Apply the Gram-Schmidt process to construct the *QR Factorization* of any $m \times n$ matrix A whose columns are linearly independent (so $\mathrm{rank}\, A = n$). In $A = QR$, the columns of Q will be an ON basis for $\mathrm{Col}\, A$, and R will be upper-triangular.

QR Factorization

Theorem (QR Factorization)

Let $A \in \mathbb{R}^{m\times n}$ have rank n (lin indep cols). Then A can be factored as $A = QR$, where the columns of $Q \in \mathbb{R}^{m\times n}$ form an ON basis for $\operatorname{Col} A$, and $R \in \mathbb{R}^{n\times n}$ is UT with pos diag entries.

$$A = \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{10}} \\ \frac{1}{2} & 0 & \frac{-2}{\sqrt{10}} \\ \frac{1}{2} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{2} & \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{10}} \end{bmatrix} = Q.$$

Since $Q^TQ = I_3$, $A = QR \implies Q^TA = Q^TQR = R$. So

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{14}} & 0 & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & \frac{14}{\sqrt{14}} & \frac{-2}{\sqrt{14}} \\ 0 & 0 & \frac{5}{\sqrt{10}} \end{bmatrix}$$

Proof of QR Decomposition

Theorem (QR Factorization)

Let $A \in \mathbb{R}^{m \times n}$ have rank n (lin indep cols). Then A can be factored as $A = QR$, where the columns of $Q \in \mathbb{R}^{m \times n}$ form an ON basis for $\mathrm{Col}\, A$, and $R \in \mathbb{R}^{n \times n}$ is UT with pos diag entries.

Proof: Key property of basis constructed by Gram-Schmidt: $\mathrm{Span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\} = \mathrm{Span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}\ \forall\, k \in [n]$. Hence, $\exists r_{ik}$ s.t. $\vec{x}_k = r_{1k}\vec{u}_1 + \cdots + r_{kk}\vec{u}_k + 0\vec{u}_{k+1} + \cdots + 0\vec{u}_n$. WLOG, $r_{kk} \geq 0$ (or change sign of $\vec{u}_k$). Set $\vec{r}_k = (r_{1k}, \ldots, r_{kk}, 0, \ldots, 0) \in \mathbb{R}^n$. Then above says $Q\vec{r}_k = \vec{x}_k$.Set

$R = [\vec{r}_1\ \ \ldots\ \ \vec{r}_n]$, get $A = [\vec{x}_1\ \ \ldots\ \ \vec{x}_n] = [Q\vec{r}_1\ \ \ldots\ \ Q\vec{r}_n] = QR$.

Q1: Why is R UT?

Q2: Why is R invertible?

Q3: Why are all diagonal entries > 0 (not just ≥ 0)?