Outline & Objectives

- Define the notion of *similar matrices* and show that similar matrices have the same eigenvalues.

- Define the *(algebraic) multiplicity* of an eigenvalue.
Definition (Similar matrices)

Call two square matrices \( A, B \in \mathbb{R}^{n \times n} \) similar if \( \exists \) an invertible \( P \in \mathbb{R}^{n \times n} \) such that \( A = PBP^{-1} \); then write \( A \sim B \).

\[
A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/7 & 1/7 \\ 5/7 & -2/7 \end{bmatrix}.
\]

Proposition (Similarity is an equivalence relation)

\( \forall A, B, C \in \mathbb{R}^{n \times n}, \) we have (1) \( A \sim A \) (reflexive);
(2) \( A \sim B \implies B \sim A \) (symmetric);
(3) \( A \sim B \) and \( B \sim C \implies A \sim C \) (transitive);

Why is it useful that \( A \) is similar to a diagonal matrix? Say \( A \) is an monthly update matrix, and want to project out ten years.\n
\[
A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}.
\]

Similarly \( A^n = PD^nP^{-1} \), so \( A^{120} = P \begin{bmatrix} 6^{120} & 0 \\ 0 & (-1)^{120} \end{bmatrix} P^{-1} \).
Similar matrices have the same eigenvalues

**Theorem**

If \( A \sim B \), then \( A \) and \( B \) have the same characteristic polynomial, hence the same eigenvalues (with the same multiplicities).

**Proof:** \( \exists P \) invertible such that \( A = PBP^{-1} \). Thus,

\[
|A - \lambda I| = |PB^{-1} - \lambda PI^{-1}| = |P(B - \lambda I)P^{-1}| = \\
|P| \cdot |B - \lambda I| \cdot |P^{-1}| = |P| \cdot \frac{1}{|P|} \cdot |B - \lambda I| = |B - \lambda I|.
\]

\[
A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} = B.
\]

**Definition (Multiplicity of an eigenvalue \( \lambda \))**

The \textit{(algebraic) multiplicity} of an eigenvalue \( \lambda \) of \( A \), is its multiplicity as a root of \( \chi(\lambda) \).

Suppose an \( 11 \times 11 \) matrix \( A \) has \( \chi(\lambda) = -\lambda^{11} + \lambda^{7} - \lambda^{3} \).