Video Lecture F12: Singular Value Decomposition 1

Tom Roby

Outline & Objectives

- MAIN GOAL: Demonstrate the existence of a *singular value decomposition (SVD)* for any $A \in \mathbb{R}^{m \times n}$, and analyze how it relates to earlier work.
- We can only diagonalize *some* $A \in \mathbb{R}^{n \times n}$ to get $A = PDP^{-1}$, and only orthogonally diagonalize *symmetric* $A \in \mathbb{R}^{n \times n}$. By contrast, the SVD factors *any rectangular* $m \times n$ matrix as $A = U\Sigma V$, with U, V orthogonal (when A is real), and Σ (block) diagonal.
- The positive (diagonal) entries of Σ in the SVD, $\sigma_1, \dots, \sigma_r$ are the *singular values*; they are the square roots of the eigenvalues of $A^T A$, not of A itself.

Max stretch and singular values

$A \in \mathbb{R}^{n\times n}$ & $A\vec{x} = \lambda\vec{x} \Rightarrow \|A\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\cdot\|\vec{x}\| = |\lambda|$ if $\|\vec{x}\| = 1$.

How to maximize stretch for rectangular A? $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

$\|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^T A\,\vec{x}$. Maximize subject to $\|\vec{x}\| = 1$!

$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ has $\lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0$, corr to

$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$.

$A\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, & $A\vec{v}_2 = \frac{1}{\sqrt{18}}\begin{bmatrix} 9 \\ -9 \end{bmatrix} = 3\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Definition

The singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ *of* $A \in \mathbb{R}^{m\times n}$ *are given by* $\sigma_i := \sqrt{\lambda_i}$, *where* $\lambda_i \in \operatorname{Spec} A^T A$.

Singular Value Decomposition

Theorem (Orthogonal basis for $\operatorname{Col} A$)

For $A \in \mathbb{R}^{m\times n}$, say $A^T A$ has ON basis of eigenvectors $\{\vec{v}_1, \dots, .\vec{v}_n\}$, with corr $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and suppose A has exactly r nonzero (positive) singular values. Then $\{A\vec{v}_1, \dots, .A\vec{v}_r\}$ is an orthogonal basis for $\operatorname{Col} A$ and $\operatorname{rank} A = r$.

Proof: For $i \neq j$, $(A\vec{v}_i)^T(A\vec{v}_j) = v_i^T A^T A\, \vec{v}_j = \vec{v}_i^T \lambda_j \vec{v}_j = \lambda_j \cdot 0 = 0$.

Now, $A\vec{v}_j = \vec{0} \iff 0 = \|A\vec{v}_j\| = \sigma_j$. Thus, $\{A\vec{v}_1, \dots, .A\vec{v}_r\}$ is lin indep. It clearly spans $\operatorname{Col} A$, so it's a basis. ■

Theorem (Singular Value Decomposition)

For any $A \in \mathbb{R}^{m\times n}$, we can find $U \in \mathbb{R}^{m\times m}$ orthogonal, $V \in \mathbb{R}^{n\times n}$ orthogonal and $\Sigma \in \mathbb{R}^{m\times n}$ such that $A = U\Sigma V^T$, where

$\Sigma = \begin{bmatrix} D & 0_{r\times n-r} \\ 0_{m-r\times r} & 0_{m-r\times n-r} \end{bmatrix}$ *and the positive singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are diag entries of D.*

Equivalently, $AV = U\Sigma \implies A\vec{v}_i = \sigma_i \vec{u}_i$ for $i \in [r]$.

Computing the SVD

Want: $A = U\Sigma V^T$, where $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. E.g., $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

(1) Compute (orthog) diagonalization of $A^T A$, giving v_i and λ_i:

$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ has $\lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0$, corr to

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}.$$

(2) Construct V and Σ: $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}$ & $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$.

(3) Construct U, starting with normalizations of $A\vec{v}_1 \cdots A\vec{v}_r$:

$$A\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \& \ A\vec{v}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 9 \\ -9 \end{bmatrix} = 3 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

$$\implies U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$