Video Lecture F10: Classifying Quadratic Forms

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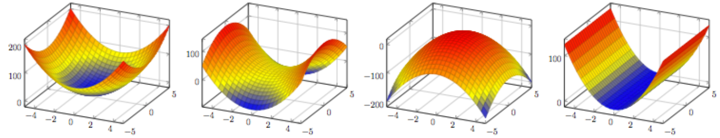
Outline & Objectives

- Apply the Principal Axes Theorem to classify quadratic forms based on their eigenvalues: The "sign" of a quadratic form (positive definite, negative definite, indefinite) is determined by the sign of its *spectrum* (i.e., its eigenvalues);

Classifying Quadratic Forms

Pictured below are graphs of the quadratic forms
(a) $5x_1^2 + 3x_2^2$, (b) $5x_1^2 - 3x_2^2$, (c) $-5x_1^2 - 3x_2^2$, (d) $5x_1^2$.



Definition

Call a quadratic form $\mathcal{Q}$:

1. *positive definite if $\mathcal{Q}(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$;*
2. *positive semidefinite if $\mathcal{Q}(\vec{x}) \geq 0$ for all $\vec{x} \neq \vec{0}$;*
3. *negative definite if $\mathcal{Q}(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$;*
4. *negative semidefinite if $\mathcal{Q}(\vec{x}) \leq 0$ for all $\vec{x} \neq \vec{0}$; and*
5. *indefinite if $\mathcal{Q}$ assume both positive and negative values.*

Theorem

For symmetric $A \in \mathbb{R}^{n\times n}$*, the QF* $Q(\vec{x}) = \vec{x}^\top A\vec{x}$ *is:*

1. *pos definite* $\iff$ *all the eigenvalues of* A *are* > 0.
2. *pos semidefinite* $\iff$ *all the eigenvalues of* A *are* ≥ 0.
3. *neg definite* $\iff$ *all the eigenvalues of* A *are* < 0
4. *neg semidefinite* $\iff$ *all the eigenvalues of* A *are* ≤ 0.
5. *indefinite* $\iff$ A *has both pos and neg eigenvalues.*

Proof: By the Principal Axes Theorem $\exists$ an orthogonal change of variables $\vec{x} = P\vec{y}$ such that $\mathcal{Q}(\vec{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \ldots$

Classify the QF $\mathcal{Q} = x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3$.