Video Lecture E7: Matrix-vector Equations

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Outline & Objectives

- Interpret (a) systems of linear equations as expressing (b) a *single* Matrix-vector equation or (c) one vector lies in the span of several others.

Matrix-vector multiplication

Definition

Define the product of an $m \times n$ matrix A, whose columns are $\vec{a}_1, \ldots, \vec{a}_n$, with a vector $\vec{x} \in \mathbb{R}^n$ by

$$A\vec{x} = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n.$$

$$\begin{bmatrix} 3 & -1 & 2 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

$\begin{cases} 3x_1 - x_2 + 2x_3 = 7 \\ -2x_1 \qquad + 3x_3 = 9 \end{cases}$ has solution $\vec{x} = (0, -1, 3)$. So does $A\vec{x} = \vec{b}$. Let's try $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Three ways to view a linear system

Theorem

Let $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$ *be an* $m \times n$ *matrix,* $\vec{b} \in \mathbb{R}^m$. *TFAE*

(1) the solutions $\vec{x} \in \mathbb{R}^n$ *to* $A\vec{x} = \vec{b}$ *(The Following Are Equal)*

(2) the solutions to $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$

(3) the solutions to augmented matrix $\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$

Q: When is there a solution? That's the next video!

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 4 & 5 & -2 \\ 3 & 6 & 7 & -2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$