Video Lecture E3: Echelon form

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Outline & Objectives

- Systematically compute solutions to linear equations by reducing the corresponding matrices to an **echelon form**.

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RECALL: The three *elementary row operations* are

1. *Replacement:* Add a multiple of one row to another row;
2. *Interchange:* Interchange two rows; and
3. *Scaling:* Multiply one row by a nonzero constant.

Review of example from Video Lecture E2

$$\begin{cases} \quad\quad y + 2z = 3 \\ 2x \quad\quad - 6z = -8 \\ 3x + 6y - 2z = -4 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 2 & 0 & -6 & -8 \\ 3 & 6 & -2 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

This is now in *echelon form*

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This is now in *reduced echelon form*

Echelon Form

Definition

A matrix M is in (row) echelon form if it satisfies:

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zeros. (Actually redundant)*

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M is in **reduced** *(row) echelon form if it also satisfies:*

4. *The leading entry in each nonzero row is a 1.*
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$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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EG: $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

New Frame

EG: Solve $\begin{cases} x_1 \phantom{{}+x_2} + x_3 + 3x_4 = 1 \\ 3x_1 + x_2 - x_3 + 4x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 3 \end{cases}$