Video Lecture D3: Determinants, column operations, & products

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Outline & Objectives

- Analyze how column operations affect the $\det A$ via $\det A^T = \det A$.
- Prove that $\det AB = (\det A)(\det B)$ and apply it in theory and practice.

Taking transposes leaves $\det A$ unchanged

Theorem (Determinant of a transpose)

For any $A \in \mathbb{R}^{n\times n}$, $\det A^T = \det A$.

Proof: $\because$ Laplace expansion works along any row or any column. ■

Corollary

Let $A \in \mathbb{R}^{n\times n}$.

1. *Rescaling a column of A by k rescales $\det A$ by k.*
2. *Interchanging two columns of A changes the sign of $\det A$.*
3. *Adding a multiple of one col to another leaves $\det A$ unchanged.*

$$\begin{vmatrix} -1 & 5 & 2 \\ 3 & 1 & -6 \\ 2 & 6 & -4 \end{vmatrix} = \begin{vmatrix} -1 & 5 & 0 \\ 3 & 1 & 0 \\ 2 & 6 & 0 \end{vmatrix} = 0.$$

Determinants of Matrix Products

Theorem (Determinant of product is product of dets)

For any two (square) $A, B \in \mathbb{R}^{n\times n}$, $\det AB = (\det A)(\det B)$.

Proof Sketch: *Claim:* $\det A = 0 \implies \det AB = 0$.
$\because$ A not invertible $\implies$ AB not invertible. (Similar if $\det B = 0$.)
Claim: If E is elementary matrix, then $\det EB = (\det E)(\det B)$.
If A invertible, then $A \sim I_n \implies A = E_\ell E_{\ell-1} \cdots E_1 I_n$. Then
$|AB| = |E_\ell \cdots E_1 B| = |E_\ell| \cdot |E_{\ell-1} \cdots E_1 B| = \cdots =$
$|E_\ell| \cdots |E_1||B| = |E_\ell \cdots E_1|B| = |A||B|$. ■

Q1: Suppose $\det A^5 = 0$. Can A be invertible?

Q2: How does $\det(AB)$ compare with $\det(BA)$?

Q3: If $U \in \mathbb{R}^{n\times n}$ satisfies $UU^T = I$, what can we say about $\det U$?