Memorize the inductive definition of the determinant of a square matrix and compute examples.

Generalize this to the Laplace expansion, which computes det A by cofactors, then analyze how to simplify computations via the choice of which row or column to expand over.

Leverage this to demonstrate the simple formula for the determinant of a triangular matrix.
Inductive definition of determinants

Definition (Determinant of a square matrix)
For $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, let $A_{k\ell} \in \mathbb{R}^{n-1 \times n-1}$ be $A$ with the $k$th row and $\ell$th column of $A$ deleted. Set $\det[a] = a$ and for $k \geq 2$

\[ |A| = \det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n}a_{1n} \det A_{1n}. \]

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 4 \\
-1 & 2 & 3
\end{bmatrix}
\]

Definition (Cofactor)
The $(i, j)$-cofactor of $A$ is $C_{ij} := (-1)^{i+j} \det A_{ij}$.

Theorem (Laplace Expansion)
\forall k, \ell, \det A = a_{k1}C_{k1} + \cdots + a_{kn}C_{kn} = a_{1\ell}C_{1\ell} + \cdots + a_{n\ell}C_{n\ell}.

Can compute $\det A$ by expanding along any row or column!
Triangular matrices

How should we expand

\[
\begin{bmatrix}
2 & 4 & 1 & 3 \\
0 & 3 & -1 & -2 \\
0 & 0 & -2 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]?

Definition

Call a matrix \( A = [a_{ij}] \) upper triangular if \( a_{ij} = 0 \) for \( i > j \) and lower triangular if \( a_{ij} = 0 \) for \( i < j \). Call \( A \) triangular either way.

Theorem

For a triangular matrix \( A \), \( \det A = a_{11}a_{22} \ldots a_{nn} \) (product of its diagonal entries).