# Video Lecture B9: Vector Space Dimension

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# Outline & Objectives

- Define the *dimension* of a vector space, an intrinsic invariant that captures the notion of its "size".
- Compute the dimension of several familiar vector spaces.

# Dimension of a vector space

**Lemma**

*Assume $V$ has a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Then any set in $V$ with more than $n$ vectors is linearly dependent.*

*Pf:* If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ with $p > n$, then $\{[\vec{u}_1]_{\mathcal{B}}, \ldots, [\vec{u}_p]_{\mathcal{B}}\} \subseteq \mathbb{R}^n \ldots$

**Theorem (Dimension Theorem)**

*If $V$ has a basis $\mathcal{B}$ of cardinality $n$, then every basis of $V$ has cardinality $n$.*

*Proof:* Let $\mathcal{C}$ be a basis of $V$. Since $\mathcal{C}$ is linearly independent, $\#\mathcal{C} \leq \#\mathcal{B}$ by lemma. Symmetrically, since $\mathcal{B}$ is linearly independent, $\#\mathcal{B} \leq \#\mathcal{C}$. Hence, $\#\mathcal{C} = \#\mathcal{B} = n$. ■

**Definition**

*If $V$ is not spanned by any finite set of vectors, then we call $V$ infinite dimesional. Otherwise, $V$ has a finite basis $\mathcal{B}$ and we define the dimension of $V$ by $\dim V := \#\mathcal{B}$. Set $\dim\{\vec{0}\} = 0$.*

## Examples

- $\dim \mathbb{R}^n = n$.
- $\dim \mathbb{R}^{m\times n} = mn$.
- $\dim \mathbb{P}_n = n+1$.
- $\dim \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\} = 1$
- $\dim \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ -6 \end{bmatrix} \right\} = 2$
- $\dim \left\{ \begin{bmatrix} 2a-4c \\ a+b-2c \\ 3b \end{bmatrix} : a, b, c \in \mathbb{R} \right\} =$
- What are all possible subspaces of $\mathbb{R}^3$? of $\mathbb{R}^4$?