Video Lecture B7: Coordinate Systems

Tom Roby

Outline & Objectives

- Demonstrate that every vector has a *unique* representation in terms of a basis $\mathcal{B}$ for a vector space V.
- Harness this to give *coordinates* for any given vector relative to $\mathcal{B}$. When V already has a standard basis (e.g., $V = \mathbb{R}^n$, and compute the corresponding *change of coordinate matrix* $P_{\mathcal{B}}$.

The Unique Representation Theorem

Theorem (Unique Representation Theorem)

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for V. Then each $\vec{x} \in V$ has a unique set of scalars $\{c_i\}$ such that $\vec{x} = c_1\vec{b}_1 + \cdots c_n\vec{b}_n$ (1).

Proof: We can find $\{c_i\}$ satisfying (1) since $\mathcal{B}$ spans V.
Why unique? Suppose $\vec{x} = d_1\vec{b}_1 + \cdots d_n\vec{b}_n$ as well...

Definition (coordinates)

Call the above weights $c_1, \ldots, c_n$ the $\mathcal{B}$-coordinates of $\vec{x}$. Write

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

(coordinate vector); call $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ the coordinate mapping (determined by $\mathcal{B}$).

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$. Say $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$; what is $\vec{x}$?

Change of Coordinates matrix

For $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ got $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$. In general,

Definition (change of coordinate matrix in $\mathbb{R}^n$)

Let $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$. Then $\vec{x} = P_{\mathcal{B}}[x]_{\mathcal{B}}$, where $P_{\mathcal{B}}$ is the change of coordinate matrix (from $\mathcal{B}$ to $\mathcal{E}$.) (Really $[\vec{x}]_{\mathcal{E}} = P_{\mathcal{B}}[x]_{\mathcal{B}}$.)

How to express e.g., $\begin{bmatrix} 0 \\ -1 \end{bmatrix}_{\mathcal{E}}$ in terms of $\mathcal{B}$?

Why is $P_{\mathcal{B}}$ invertible in general?

So $[\vec{x}]_{\mathcal{E}} = P_{\mathcal{B}}[x]_{\mathcal{B}} \iff [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}[x]_{\mathcal{E}}$.

What is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{E}}$ in terms of $\mathcal{B}$-coordinates?