Video Lecture B6: Bases for the nullspace and column space of a matrix

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Outline & Objectives

- Compute bases for the nullspace and for the column space of a matrix A.

Bases for the Column Space

$$A = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 2 & 5 & -8 & 1 \\ 1 & 4 & -7 & 5 \\ 2 & 4 & -6 & -1 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-\vec{b}_1 + 2\vec{b}_2 + \vec{b}_3 = 0$, equivalently $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ is a solution to $B\vec{x} = 0$;

hence, also a soln to $A\vec{x} = 0$.

Key: Row ops preserve linear dep. relations! So a set of columns of A is linearly (in)dependent $\iff$ same columns of B are.

Proposition

The pivot columns of a matrix A form a basis of $\mathrm{Col}\, A$

Need to row-reduce A to figure out pivots, but *must* use pivot columns of *original* matrix A.

Bases for the Nullspace

We already know how to do this!

$$A = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 2 & 5 & -8 & 1 \\ 1 & 4 & -7 & 5 \\ 2 & 4 & -6 & -1 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ So

$$\begin{cases} x_1 = -x_3 \\ x_2 = 2x_3 \\ x_4 = 0 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$ So

$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathrm{Nul}\, A = \mathrm{Nul}\, B$.

Although $\mathrm{Nul}\, A = \mathrm{Nul}\, B$, in general $\mathrm{Col}\, A \neq \mathrm{Col}\, B$!.