Video Lecture B5: Vector space bases

Tom Roby

Outline & Objectives

- Generalize the notion of *linear independence* to (general) vector spaces, and define *a basis* of a vector (sub)space.
- Prove that any spanning set for a vector (sub)space H contains a basis for H.

Linear (in)dependence and bases in vector spaces

Definition (linear independence)

A set of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ *is called linearly independent if the vector equation* $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ *has only the trivial soln. (all* $x_i = 0$*). The set is linearly dependent if there exists a nontrivial solution,* c_i *not all zero, such that* $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$*. [*$\rightsquigarrow$ *linear dependence relation] (Same as the def in* $\mathbb{R}^n$*!)*

Proposition

(1) A set containing the zero vector $\vec{0}$ *is linearly dependent.*
(2) $S = \{\vec{v}_1, \vec{v}_2\}$ *is linearly dependent* $\iff$ *one of* $\vec{v}_1, \vec{v}_2$ *is a multiple of the other.*

Definition (basis of a vec space)

Let H *be a subspace of a vector space* V*. An indexed set* $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ *is a basis of* H *if: (i)* $\mathcal{B}$ *is linearly independent (ii)* $\mathrm{Span}\,\mathcal{B} = H$*.*

Examples

Classify each set as either linearly independent or dependent:

1. $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \right\}$ in $\mathbb{R}^{2\times 2}$
2. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathbb{R}^{2\times 2}$
3. $\{2 + t^3, 1 - 3t + t^2, 1 + 2t\}$ in $\mathbb{P}_3$
4. $\{\sin^2 x,\ \cos^2 x,\ \cos 2x\}$ in $\mathcal{C}[-\pi, \pi]$
5. The columns of the matrix $\begin{bmatrix} 3 & -4 & 7 & 17 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{R}^4$.

Which are bases?

Linear Independence in Theory

Theorem

An indexed set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of $p \geq 2$ vectors is linearly dependent $\iff$ at least one vector is a linear combination of the others. In fact, if $\vec{v}_1 \neq 0$, then there is $j \geq 2$ such that $v_j = d_1\vec{v}_1 + \cdots + d_{j-1}\vec{v}_{j-1}$, some $d_i \in \mathbb{R}$.

The proof is the same as in VL#E12 for $\mathbb{R}^n$.

Theorem (Spanning Set Theorem)

(i) Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ and $H = \operatorname{Span} S$. If $\vec{v}_k$ is a linear combination of $S' := S \setminus \{\vec{v}_k\}$, then S' also spans H.
(ii) If $H \neq \{\vec{0}\}$, then some subset of S is a basis for H.

Proof: By rearranging vectors in S, WLOG assume $\vec{v}_k = \vec{v}_p$. Then can write $\vec{v}_p = a_1\vec{v}_1 + \cdots + a_{p-1}\vec{v}_{p-1}$. Now if $\vec{x} \in H = \operatorname{Span} S$, then $\vec{x} = c_1\vec{v}_1 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p$

$$= (c_1 + c_pa_1)\vec{v}_1 + \cdots + (c_{p-1} + c_pa_{p-1})\vec{v}_{p-1} \in \operatorname{Span} S'.$$