Video Lecture B10: Dimension of Vector Subspaces

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Outline & Objectives

- Analyze how any linearly independent set can be extended to bases (by adding elements), just as spanning sets can be reduced to bases (by removing elements).
- Deduce that any set $\mathcal{S}$ with $\#\mathcal{S} = \dim V$ will be a basis if either (1) it is linearly independent or (2) it spans V.
- Relate the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ to the number of free variables and pivots of a matrix A.

Bases for Subspaces of FDVS (Fin Dim Vec Space(s))

Recall the Spanning Set Theorem from an earlier lecture.

Lemma

Let H be a subspace of a FDVS V, and let $\mathcal{S} \subseteq H$ be lin indep. Then $\mathcal{S}$ can be extended to a basis for H. Hence, $\dim H \leq \dim V$.

Pf: Let $\mathcal{S} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be lin indep. If $\operatorname{Span} \mathcal{S} = H$, we're done. Otherwise, $\exists\, \vec{u}_{k+1} \in H$ but *not* in $\operatorname{Span} \mathcal{S}$. So $\mathcal{S}' := \{\vec{u}_1, \ldots, \vec{u}_k, \vec{u}_{k+1}\}$ is lin. indep. ...■

Theorem (Basis Theorem)

Let $\dim V = n$. Then (1) any lin indep set with n elements is a basis, and (2) any spanning set with n elements is a basis.

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}. \ \mathcal{R} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ 2 \end{bmatrix} \right\}.$$

Dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$

$$A = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 2 & 5 & -8 & 1 \\ 1 & 4 & -7 & 5 \\ 2 & 4 & -6 & -1 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem

For any $A \in \mathbb{R}^{m \times n}$:
(1) $\dim \operatorname{Col} A =$ the number of pivot columns of A.
(2) $\dim \operatorname{Nul} A =$ the number of free variables of A.

$$C = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & -1 & 2 & 6 \\ 2 & 2 & 4 & 4 \end{bmatrix} \sim D = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$