Math 3250 (Roby) **Practice Midterm #2** Spring 2013

Closed book, notes (except reading notes), calculators, computers, cell phones, etc. Make sure to show your reasoning so I can give partial credit.

1. A line of snakes is waiting for a bus. The length of the line is n meters. Each snake is k meters long for some $k \geq 2$. A snake that is k meters long has $k-2$ spots, each either black or orange. Also, each snake is either very poisonous, slightly poisonous, or not poisonous at all. Find a simple formula for the number $f(n)$ of possibilities. For instance, $f(1) = 0$ (since no snake is one meter long), $f(2) = 3$ (one snake of length 2 meters with no spots and three possible levels of poisonosity). Make sure to use that the snakes are waiting in a *line*. Snakes with the same characterstics are indistinguishable, except for their position in the line.

   This is an application of Thm. 8.13. For a k-meter snake there are $3 \cdot 2^{k-2}$ possibilities. Hence
   $$A(x) = \sum_{k \geq 2} 3 \cdot 2^{k-2} x^k = \frac{3x^2}{1-2x}.$$
   Now from Thm. 8.13 we get
   $$\begin{aligned}\sum_{n\geq 0} f(n)x^n &= \frac{1}{1-A(x)} \\ &= \frac{1}{1-\frac{3x^2}{1-2x}} \\ &= \frac{1-2x}{1-2x-3x^2} \\ &= \frac{1-2x}{(1-3x)(1+x)} \\ &= \frac{1/4}{1-3x} + \frac{3/4}{1+x} \\ &= \frac{1}{4}\sum_{n\geq 0} 3^n x^n + \frac{3}{4}\sum_{n\geq 0}(-1)^n x^n\,.\end{aligned}$$
   Hence, $f(n) = \frac{1}{4}\left(3^n + 3\cdot(-1)^n\right)$.

2. There are n (distinguishable) fish in an aquarium. The person who feeds the fish gives an odd number of the fish either a red pollywog or a turquoise pollywog to eat. He gives an odd number of the fish either a black sea anemone, a purple sea anemone, or a green sea anemone to eat. The poor remaining fish get nothing to eat. (No fish gets more than one item.) For instance, $f(1) = 0$ (since there must be at least one fish that gets a pollywog and at least one fish that gets a sea anemone) and $f(2) = 12$ (two choices for which fish gets a pollywog, two choices for the pollywog color, and three choices for the sea anemone color). Find a simple formula for $f(n)$ not involving any summation symbols.

This is analogous to problem ($E10$) and uses Thm. 8.21 on products of EGF. For fish who get a pollywog, the EGF is

$$A(x) = \sum_{n\geq 0} 2^{2n+1}\frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

For fish who get a sea anemone, the EGF is

$$B(x) = \sum_{n\geq 0} 3^{2n+1}\frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}\left(e^{3x} - e^{-3x}\right)$$

For the unfortunate remaining fish:

$$C(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x$$

Hence,

$$\begin{aligned}\sum_{n\geq 0} f(n)\frac{x^n}{n!} &= A(x)B(x)C(x)\\ &= \frac{1}{2}\left(e^{2x} - e^{-2x}\right)\frac{1}{2}\left(e^{3x} - e^{-3x}\right)e^x\\ &= \frac{1}{4}\left(e^{6x} - e^{2x} - 1 + e^{-4x}\right).\end{aligned}$$

whence $f(n) = \frac{1}{4}\left(6^n - 2^n + (-4)^n\right)$, for $n \geq 1$.

3. How many positive integers are there less than or equal to a million that are neither perfect squares, perfect cubes, nor perfect fourth powers?

   Note that eliminating perfect squares eliminates perfect fourth powers automatically, so we only need to think about squares and cubes here: A simple application of PIE yields

$$10^6 - 10^3 - 10^2 + 10^1 = 998,910\,.$$

4. Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x)$ and use this to find a closed form expression for r_n itself.

   Carefully taking into account the initial terms, the recursive formula for r_n yields that $R(x) - xR(x) - 6x^2R(x) = 3 + x$. Therefore,

$$R(x) = \frac{3+x}{1-x-6x^2} = \frac{1}{1+2x} + \frac{2}{1-3x}\,,$$

   where the last equality uses the method of partial fractions. Now expanding these two geometric series yields the explicit formula $r_n = (-2)^n + 2(3)^n$ for $n \geq 0$.

5. (a) Prove that the ordinary generating function for the sequence $c_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

   Use Newton's binomial theorem. For details see Bona Ex. 4.26 (and solution).

   (b) Prove that

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n .$$

   Interpret the LHS of this as the coefficient of x^n in the product of the OGF of (a) with itself, whose closed form expression is $(1-4x)^{-\frac{1}{2}} \cdot (1-4x)^{-\frac{1}{2}} = \frac{1}{(1-4x)}$.

   (c) (Extra credit) Can you give a combinatorial proof?

   This is harder than it looks...

6. For $n \geq 1$, let $f(n)$ be the number of $n \times n$ matrices of 0's and 1's such that every row and every column has at least one 1. For instance, $f(1) = 1$ and $f(2) = 7$. Use the Principle of Inclusion-Exclusion to give a formula for $f(n)$ as a single sum. [May be tricky!]

   For each $S \subseteq [n]$ let $g(s)$ be the number of $n \times n$ matrices of 0's and 1's such that every row contains a 1, and if $i \in S$, then column i does not contain a 1. Each row then has $n-k$ available positions where we can place the 1's. Thus if $\#S = k$, then there are $2^{n-k} - 1$ possibilities for each row. Hence, $g(S) = (2^{n-k} - 1)^n$. By the sieve method,

$$\begin{aligned} f(n) &= g(\emptyset) - \sum_{\#S=1} g(S) + \sum_{\#S=2} g(S) - \cdots + (-1)^n \sum_{\#S=n} g(S) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^n . \end{aligned}$$

   (The last term is 0 and can be omitted.) This problem can also be done by writing $f(n)$ as a double sum and using the binomial theorem to reduce it to a single sum, though the solution above is simpler.

7. Go over all the old homework problems, with particularly attention to anything that gave you trouble.