Math 3250 (Roby) **Practice Midterm #2** Spring 2013

**Closed book, notes (except reading notes), calcluators, computers, cell phones, etc. Make sure to show your reasoning so I can give partial credit.**

1. A line of snakes is waiting for a bus. The length of the line is $n$ meters. Each snake is $k$ meters long for some $k \geq 2$. A snake that is $k$ meters long has $k-2$ spots, each either black or orange. Also, each snake is either very poisonous, slightly poisonous, or not poisonous at all. Find a simple formula for the number $f(n)$ of possibilities. For instance, $f(1) = 0$ (since no snake is one meter long), $f(2) = 3$ (one snake of length 2 meters with no spots and three possible levels of poisonosity). Make sure to use that the snakes are waiting in a *line*. Snakes with the same characterstics are indistinguishable, except for their position in the line.

2. There are $n$ (distinguishable) fish in an aquarium. The person who feeds the fish gives an odd number of the fish either a red pollywog or a turquoise pollywog to eat. He gives an odd number of the fish either a black sea anemone, a purple sea anemone, or a green sea anemone to eat. The poor remaining fish get nothing to eat. (No fish gets more than one item.) For instance, $f(1) = 0$ (since there must be at least one fish that gets a pollywog and at least one fish that gets a sea anemone) and $f(2) = 12$ (two choices for which fish gets a pollywog, two choices for the pollywog color, and three choices for the sea anemone color). Find a simple formula for $f(n)$ not involving any summation symbols.

3. How many positive integers are there less than or equal to a million that are neither perfect squares, perfect cubes, nor perfect fourth powers?

4. Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x)$ and use this to find a closed form expression for $r_n$ itself.

5. a) Prove that the ordinary generating function for the sequence $c_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

   b) Prove that
$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n .$$

   c) (Extra credit) Can you give a combinatorial proof?

6. For $n \geq 1$, let $f(n)$ be the number of $n \times n$ matrices of 0's and 1's such that every row and every column has at least one 1. For instance, $f(1) = 1$ and $f(2) = 7$. Use the Principle of Inclusion-Exclusion to give a formula for $f(n)$ as a single sum. [May be tricky!]

7. Go over all the old homework problems, with particularly attention to anything that gave you trouble.