Math 3250 (Roby) **Practice Midterm #1** Spring 2013

Closed book, notes, calcluators, computers, cell phones, etc. Make sure to show your reasoning so I can give partial credit.

1. Let (a_i, b_i, c_i) for $i \in [9]$ be nine vectors of length three in $\mathbb{Z}^3$.

   (a) Show that (at least) two of these vectors have a sum whose coordinates are all even integers.

   Given an integer a, let $\bar{a} = a \bmod 2 = 0$ if a is even and 1 if a is odd. There are eight possible sequences $(\bar{a_i}, \bar{b_i}, \bar{c_i})$; hence, by the pigeonhole-principle, there are two values $1 \le i < j \le 9$ for which

   $$(\bar{a_i}, \bar{b_i}, \bar{c_i}) = (\bar{a_j}, \bar{b_j}, \bar{c_j})\,.$$

   Then the sum of these vectors will have even coordinates.

   (b) Show that this result is best possible, i.e., the conclusion may fail if we have only eight vectors.

   From (a) we need to choose the vectors so that the "reduced vectors" $(\bar{a_i}, \bar{b_i}, \bar{c_i})$ are all different. The simplest choice is all binary vectors of length three: $000, 001, 010, 011, 101, 110, 1$

2. Express the number $f(n)$ of partitions of the integer $n \ge 1$ with no parts equal to 1 in terms of values of the partition function, i.e., in terms of the numbers $p(1), p(2), p(3)\ldots,$ where $p(k)$ is the number of partitions of k. Your formula should be simple. For example, $f(5) = 2$, counts the two partitions (5) and $(3, 2)$.

   We claim that $f(n) = p(n) - p(n-1)$. To obtain a partition of n with at least one part equal to 1, take a partition λ of $n-1$ and adjoin a new part equal to 1. Thus there are $p(n-1)$ partition of n with at least one part equal to 1, so $f(n) = p(n) - p(n-1)$.

3. (a) Find the number $f(n)$ of binary sequences $a_1, a_2, \ldots a_n$ (so each $a_i = 0, 1$) with no two consecutive 1's. Express your answer in terms of the Fibonacci numbers (given by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$).

   We have $f(0) = 1, f(1) = 2, f(2) = 3$, suggesting that $f(n) = F_{n+2}$. To prove it, we need to show that $f(n+1) = f(n) + f(n-1)$, since we have already checked the initial conditions. If a sequence $a_1, \ldots, a_{n+1}$ with no two consecutive 1's begins $a_1 = 0$, then $a_2, \ldots, a_{n+1}$ can be any sequence of length n wit hno two consecutive 1's, of which there are $f(n)$. On the other hand, if $a_1 = 1$, then $a_2 = 0$, and there are $f(n-1)$ choices for $a_3, \ldots, a_{n+1}$. Thus, $f(n+1) = f(n) + f(n-1)$.

(b) Do the same for binary sequences $a_1, a_2, \dots a_n$ satisfying $a_1 \geq a_2 \leq a_3 \geq a_4 \leq a_5 \geq \dots$ (alternating $\geq$ and $\leq$).

Again we have $f(0) = 1, f(1) = 2, f(2) = 3$, so let us prove in this case that we have the Fibonacci recurrence. The argument is slightly trickier. If $a_1 = 1$, then choose $b_2, \dots b_{n+1}$ to be a sequence satisfying $b_2 \geq b_3 \leq b_4 \geq \dots b_{n+1}$ in $f(n)$ ways and define $a_i = 1 - b_i$

4. (a) How many different ways are there to arrange the letters in the word RIFFRAFF so that two R's are not adjacent?

Subtract the number of ways where the two R's **are** adjacent (treated as a single element of the permutation) from the total number to get

$$\frac{8!}{4!2!1!1!} - \frac{7!}{4!1!1!1!} = 4 \cdot 7 \cdot 6 \cdot 5 - 7 \cdot 6 \cdot 5$$
$$= 3 \cdot 7 \cdot 6 \cdot 5 = 630\,.$$

(b) How many compositions of 24 into any number of parts have each part divisible by three?

Any such composition is in bijection with an arbitrary composition of eight simply by dividing each part by three. So the answer is $c(8) = 2^{8-1} = 128$.

5. Let $n \geq 4$. How many permutations $\pi \in S_n$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols. (Try to check your answer for $n = 4$.)

There is only one choice for $\pi(1)$. There are then $n-3$ choices for $\pi(2)$ (anything other than 2,3, or 4). Then $n-3$ choices for $\pi(3)$ (anything other than 2, $\pi(2)$, and 4, which are all different). Then $n-3$ choices for $\pi(4)$, $n-4$ choices for $\pi(5)$, etc. This gives $(n-3)^3(n-4)! = (n-3)^2(n-3)!$ choices in all. (For $n = 4$ the formula gives 1, indicating the only such permutation is 2134.)

6. Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2}\binom{n}{k}.$$

#1: Take the binomial expansion $(1+x)^n = \sum_{k=0}^{n} x^k$, differentiate twice and divide by 2 to get

$$\frac{1}{2}n(n-1)(x+1)^{n-2} = \sum_{k-0}^{n} \frac{1}{2}k(k-1)\binom{n}{k}x^{k-2}.$$

Setting $x = 1$ we get

$$\binom{n}{2}2^{n-2} = \sum_{k=0}^{n} \binom{k}{2}\binom{n}{k}.$$

#2 The RHS counts the number of ways to choose a subset S of any size from $[n]$, then choose a two element subset T from S. But we could get the same result by choosing T first in $\binom{n}{2}$ ways, then choose an arbitrary subset of the remaining $n-2$ elements (in 2^{n-2} ways), which gives the LHS.

7. Recall that $S(n,k)$ denotes the Stirling numbers of the second kind.

   (a) Prove that for all positive integers $k < n$,

   $$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k).$$

   Let $\pi = \{B_1, \ldots B_k\}$ be a partition of $[n]$ into k subsets (as counted by the LHS). Then either n is in a singleton block, in which case deleting the block leaves a partition π' of $n-1$ into $k-1$ blocks (as counted by the first summand on the RHS); or n lies in a block with at least two elements, in which case deleting n leaves a partition π'' of $n-1$ into k blocks. Reversing the above map, we find that there is only way to add a singleton block with n to π' in the first case, but that in the second case, n could be added into any of the k existing blocks, yielding k distinct partitions of $[n]$.

   (b) Explain why the number of surjective functions $f : [n] \to [k]$ is $k! \cdot S(n,k)$

   If f is a surjective function onto $[k]$, then the collection of inverse images

   $$\{f^{-1}(1), f^{-1}(2), \ldots f^{-1}(k)\}$$

   forms a partition of $[n]$ into k blocks, but with each block labelled (distinguishable). Conversely, any partition of $[n]$ into k blocks can be considered a function $[n] \xrightarrow{f} [k]$ by simply sending all elements in the same block to the same element of $[k]$. Each of the $k!$ labelings of the blocks corresponds to a distinct function, so we have in total $k!S(n,k)$ surjective functions.