Math 3250 (Roby) **Practice Midterm #1** Spring 2013

Closed book, notes, calcluators, computers, cell phones, etc. Make sure to show your reasoning so I can give partial credit.

1. Let (a_i, b_i, c_i) for $i \in [9]$ be nine vectors of length three in $\mathbb{Z}^3$.
   (a) Show that (at least) two of these vectors have a sum whose coordinates are all even integers.
   (b) Show that this result is best possible, i.e., the conclusion may fail if we have only eight vectors.

2. Express the number $f(n)$ of partitions of the integer $n \geq 1$ with no parts equal to 1 in terms of values of the partition function, i.e., in terms of the numbers $p(1), p(2), p(3) \ldots$, where $p(k)$ is the number of partitions of k. Your formula should be simple. For example, $f(5) = 2$, counts the two partitions (5) and $(3, 2)$.

3. (a) Find the number $f(n)$ of binary sequences $a_1, a_2, \ldots a_n$ (so each $a_i = 0, 1$) with no two consecutive 1's. Express your answer in terms of the Fibonacci numbers (given by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$).
   (b) Do the same for binary sequences $a_1, a_2, \ldots a_n$ satisfying $a_1 \geq a_2 \leq a_3 \geq a_4 \leq a_5 \geq \ldots$ (alternating $\geq$ and $\leq$).

4. (a) How many different ways are there to arrange the letters in the word RIFFRAFF so that two R's are not adjacent?
   (b) How many compositions of 24 into any number of parts have each part divisible by three?

5. Let $n \geq 4$. How many permutations $\pi \in S_n$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols. (Try to check your answer for $n = 4$.)

6. Find a simple formula (no summation symbols) for
$$f(n) = \sum_{k=0}^{n} \binom{k}{2} \binom{n}{k}.$$

7. Recall that $S(n, k)$ denotes the Stirling numbers of the second kind.
   (a) Prove that for all positive integers $k < n$,
$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k).$$
   (b) Explain why the number of surjective functions $f : [n] \to [k]$ is $k! \cdot S(n, k)$