Math 3250 (Roby) **Practice Final Solutions** Spring 2013

Closed book, notes (except reading notes and one 4×6 inch notecard, both sides), calculators, computers, cell phones, etc. Make sure to show your reasoning so I can give partial credit.

1. Go over all the homework problems, with particularly attention to anything that gave you trouble.

2. Go over all the problems from the midterms and practice midterms, with particularly attention to anything that gave you trouble.

3. Compute the answer to each enumeration problem below. Express your answer both in a way that has mathematical meaning (e.g., a difference of binomial coefficients) and as a nonnegative integer. If a problem is a direct consequence of an entry in the twelvefold way, explain which one (e.g., "this is equivalent to putting indistinct balls in distinct boxes surjectively").

   (a) How many subsets of the set $[12] = \{1, 2, \ldots, 12\}$ contain at least one odd integer?
   $2^{12} - 2^6 = 4032.$

   (b) In how many ways can eight people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?
   $\frac{1}{2}(8-1)! = 2520.$

   (c) How many permutations $\pi : [6] \to [6]$ satisfy $\pi(1) \neq \pi(2)$?
   $5 \cdot 5!$ (or $6! - 5!$) $= 600.$

   (d) How many permutations of $[6]$ have exactly two cycles?
   $\binom{6}{1}4! + \binom{6}{2}3! + \frac{1}{2}\binom{6}{3}2!^2 = 274.$

   (e) How many partitions of $[6]$ have exactly three blocks?
   $\binom{6}{4} + \binom{6}{1}\binom{6}{2} + \frac{1}{3!}\binom{6}{2}\binom{4}{2} = 90.$

   (f) There are four men and six women. Each man marries one of the women. In how many ways can this be done?
   $(6)_4 = 360.$

   (g) Ten people split up into five groups of two each. In how many ways can this be done?
   $9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 945.$

   (h) How many compositions of 20 use only the parts 2 and 3?
   $\binom{7}{1} + \binom{8}{4} + \binom{9}{7} + \binom{10}{10} = 114.$

(i) How many partitions of 8 are there into odd parts?

Six: $71, 53, 5111, 331, 31111, 11111111$.

(j) In how many different ways can the letters of the word BOOKKEEPER be arranged if the three E's cannot appear consecutively?

$\binom{10}{3,2,2,1,1,1} - \binom{8}{2,2,1,1,1,1} = 14,1120.$

(k) How many sequences $(a_1, a_2, \ldots, a_{12})$ are there consisting of four 0's and eight 1's if no two consecutive terms are both 0's?

$\binom{8+1}{4} = 126.$

(l) A box is filled with three azure socks, three brown socks, and four chartreuse socks. Eight socks are pulled out, one at at time. In how many ways can this be done? (Socks of the same color are indistinguishable.)

$2\binom{8}{1,3,4} + 3\binom{8}{2,3,3} + \binom{8}{2,2,4} = 2660.$

(m) How many trees are there on the vertex set $[6]$? How many rooted forests?

$6^{6-2} = 1296$; $(6+1)^{6-1}$.

4. Find the number of lattice paths from $(0,0)$ to $(20,30)$ with steps $(1,0)$ and $(0,1)$ (i.e., one unit north or one unit east) that pass through the point $(8,15)$ but do not pass through $(14,23)$. Express your answers in terms of familiar combinatorial numbers covered in the course—there's no need for a final numerical answer.

$\binom{23}{8}\binom{27}{12} - \binom{23}{8}\binom{14}{6}\binom{13}{6} = 5,996,889,323,568.$

5. Use generating functions to solve the recurrence

$$f(n+1) = 3f(n) + 2^{n+1}\,, \text{ for } n \geq 0\,,$$

with initial condition $f(0) = 1$.

Let $F(x)$ be the OGF for $f(n)$. Multiply the recurrence by x^{n+1} and sum on $n \geq 0$ to get

$$\begin{aligned} F(x) - 1 &= 3xF(x) + \frac{2x}{1-2x} \\ F(x)[1-3x] &= 1 + \frac{2x}{1-2x} = \frac{1}{1-2x} \\ F(x) &= \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x} \end{aligned}$$

Hence, $f(n) = 3^{n+1} - 2^{n+1}$.

6. (a) Let $h(n)$ be the number of ways to choose a permutation $\pi \in S_n$, then for each cycle C of π either color each element of C red or blue, or else don't color any

element of C. For instance, $h(2) = 14$, with nine possibilities for the identity permutation and five for the other. Set $h(0) = 1$. Find a simple expression (without summation symbols, product symbols, or exponential functions) for the exponential generating function for $h(n)$.

Partition the set $[n]$. On each block place a cycle, then color each element red or blue, or else don't color any element. For a block of size k we have $f(k) = (k-1)!(2^k+1)$ choices. Then by the Exponential Formula we get

$$\begin{aligned}\sum_{n\geq 0} h(n)\frac{x^n}{n!} &= \exp\sum_{k\geq 1}(k-1)!(2^k+1)\frac{x^k}{k!}\\ &= \exp\sum_{k\geq 1}(2^k+1)\frac{x^k}{k}\\ &= e^{-\log(1-2x)-\log(1-x)}\\ &= \frac{1}{(1-2x)(1-x)}.\end{aligned}$$

(b) Find a simple formula for $h(n)$.

By (a) we know that the EGF for $h(n)$ is given by

$$\frac{1}{(1-2x)(1-x)} = \frac{2}{1-2x} - \frac{1}{1-x}.$$

Hence, $h(n) = n!(2^{n+1}-1)$. Notice that if you had forgotten the $n!$ you would get the wrong answer for $h(2)$.

7. (a) Suppose that a tournament T has a directed cycle. Show that T has a directed triangle (i.e., a directed cycle of length three).

Let C be a directed cycle of T of minimal length ℓ. If $\ell > 3$, then choose two vertices u and v of C that are not adjacent in C. Any orientation of the edge uv produces a cycle of length less than ℓ, contradiction. Hence, $\ell = 3$.

(b) How many tournaments on the vertex set $[n]$ have no directed triangles?

By (a), we want to count the number of tournaments on $[n]$ with no directed cycle. Such a tournament T must have a *sink* z, since if we start at any random vertex and start walking along edges in the direction of their arrows, then we either end at a sink or create a directed cycle. This sink must be *unique*: for if z' were another sink, then the edge between z and z' would create a contradiction.

Now remove $z_1 := z$ and all incident edges from T to create a tournament T_2. The same reasoning applies to T_2, so we can find its unique sink z_2. Continuing in this way gives a unique sequence of vertices $z_1, z_2, \ldots z_n$ such that $z_i \to z_j$ whenever $i < j$. Since there are $n!$ ways to label the vertices, this gives $n!$ tournaments on $[n]$ that contain no directed cycle, hence by (a) no directed triangle.

8. The Laplacian matrix L of a certain loopless graph G has eigenvalues $0, 1, 1, 2, 4, 4, 7, 11$. Compute the number of vertices, edges, and spanning trees of G.

The number of vertices, p, is the number of eigenvalues; so $p = 8$. The diagonal entries of L are the vertex degrees, so the trace is twice the number of edges. But the trace is also the sum of the eigenvalues, namely 30, so we have 15 edges. By the matrix-tree theorem we get

$$\kappa(G) = \frac{\mu_1\mu_2\cdots\mu_{p-1}}{p} = \frac{1\cdot 1\cdot 2\cdot 4\cdot 4\cdot 7\cdot 11}{8} = 308\,.$$