Here are some extra problems, labeled ($E\#$) in the HW assignments.

1. Find the number d_n of compositions into odd parts of the positive integer n. For this problem and #35, try to express your answer in terms of Fibonacci numbers.

2. Let λ be a partition with conjugate partition λ'. Show that

$$\sum_i \lfloor \lambda_{2i-1}/2 \rfloor = \sum_i \lceil \lambda'_{2i}/2 \rceil .$$

   This can be seen almost by inspection from the Young diagram of λ after certain marks are made on it. Note. The notation $\lfloor x \rfloor$ means the greatest integer $\leq x$. For instance, $\lfloor 3 \rfloor = 3$, $\lfloor 3/2 \rfloor = 1$. Similarly, $\lceil x \rceil$ means means the least integer $\geq x$. For instance, $\lceil 3 \rceil = 3$, $\lceil 3/2 \rceil = 2$.

3. Show by simple combinatorial reasoning and induction that the Bell number $B(n)$ is even if and only if $n-2$ is divisible by 3. (**Hint:** First show combinatorially that $B(n) - B(n-1) - B(n-2)$ must be even.)

4. This problem is a variation of a result discussed in class. We have the same 100 contestants and similar rules. As before, the contestants are brought one at a time into a room with 100 boxes labeled $1, 2, , 100$. Inside each box is the name of a contestant, a different name in each box. This time a contestant must open 99 boxes, one at a time. If the contestant encounters his own name, then all contestants lose. The contestants may talk together before the first contestant enters the room with the boxes. After that there is no further communication. A contestant cannot leave a signal in the room. All boxes are closed before each contestant enters the room. Clearly the probability that the contestants win cannot exceed $1/100$, since that is the probability that the first contestant does not encounter his name. What strategy maximizes the probability that the contestants win, and what is this maximum probability?

5. Let $n \geq 3$. Pick a permutation π of $1, 2, ..., n$ at random (uniform distribution, i.e., all permutations are equally likely). What is the probability that 1, 2, and 3 are all in different cycles of π?

6. Call two permutations $a_1, ..., a_n$ and $b_1, ..., b_n$ of $1, 2, ..., n$ *equivalent* if one can be obtained from the other by switching adjacent terms that differ by at least two. For instance 254613 is equivalent to 542361, one sequence of switches being 254613, 524613, 542613, 542631, 542361. Clearly this is an equivalence relation. (It is assumed that you know about equivalence relations and equivalence classes.) How many equivalence classes are there? For instance, when $n = 3$ there are the four classes $\{123\}$, $\{132, 312\}$, $\{213, 231\}$, and $\{321\}$.

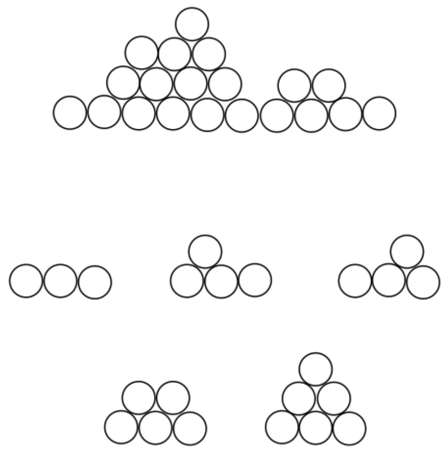

Figure 1: Stacking Pennies

7. Let $f(n)$ be the number of ways to stack pennies against a flat wall as follows: the bottom level consists of a row of n pennies, each tangent to its neighbor(s). A penny may be placed in a higher row if it is supported by two pennies below it. See Figure 1 for an example for $n = 10$ and for all five possibilities when $n = 3$. Show that $f(n) = C_n$, a Catalan number.

8. (*) Show that the number of partitions of n for which no part appears exactly once is equal to the number of partitions of n for which every part is divisible by 2 or 3. For instance, when $n = 6$ there are four partitions of the first type $(111111, 2211, 222, 33)$ and four of the second type $(222, 33, 42, 6)$. Use generating functions.

9. Show that the number of partitions of n for which no part appears more than twice is equal to the number of partitions of n for which no part is divisible by 3. For instance, when $n = 5$ there are five partitions of the first type $(5, 41, 32, 311, 221)$ and five of the second type $(5, 41, 221, 2111, 11111)$. Use generating functions.

10. Let $f(n)$ be the number of ways to paint n giraffes either red, blue, yellow, or turquoise, such that an odd number of giraffes are red and an even number are blue. Use exponential generating functions to find a simple formula for $f(n)$. (It is allowed to have no giraffes painted blue, yellow, or turquoise.)

11. Let $f(n)$ be the number of ways to partition an n-element set, and then to choose a nonempty subset of each block of the partition. Find a simple formula (no infinite sums) for the exponential generating function $G(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}$.

12. Give a simple reason why a 9-vertex simple graph cannot have the degrees of its vertices equal to 8, 8, 6, 5, 5, 4, 4, 3, 1.

13. (*) Give an example of a simple graph with exactly three automorphisms. Note that the graph K_3 (a triangle) has six automorphisms.

14. (*) Let G_n denote the complete graph K_{2n} with n vertex-disjoint edges (i.e., a complete matching) removed. Use the Matrix-Tree theorem to find $\kappa(G_n)$, the number of spanning trees of G_n. (This is not so easy. Find the eigenvalues of the Laplacian matrix L of G_n. Several tricks are needed.)

15. Let m and n be two positive integers. Find the number of Hamiltonian cycles of the complete bipartite graph $K_{m,n}$. (There will be two completely different cases.)

16. (a) Let G be the *infinite* graph whose vertices are the points (i, j) in the plane with integer coordinates, and with an edge between two vertices if the distance between them is one. Is G bipartite?

    (b) (*) What if the vertices are the same as (a), but now there is an edge between two vertices if the distance between them is an odd integer?

17. Let G be a bipartite graph for which a maximum matching has n edges. What is the smallest possible size of a maximal matching? (You need to give an example of this size and prove that no smaller size is possible in any bipartite graph for which a maximum matching has n edges.)

18. Let G and H be finite graphs. Let K consist of the union of G and H, with an edge e of G identified with an edge f of H. (Thus if G has q edges and H has r edges, then K has $q + r - 1$ edges.) Express the chromatic polynomial of K in terms of those of G and H.

19. Let $n \geq 4$. Suppose that P is a convex polyhedron with one n-vertex face and two additional vertices. What is the most number of faces that P can have?

20. Show that the complete graph K_7 can be embedded on a torus without crossing edges. Draw the torus as a square (or rectangle) with opposite edges identified.

21. (*) Find a triangle-free graph (i.e., no three vertices that are pairwise adjacent) with chromatic number four. Try to use as few vertices as possible.