Algebraic Combinatorics **HW #3** **DUE: 1 October 2013**

From the text

Ch. 2: #4, 5, 6; Ch. 3: #1 Please write careful solutions to these problems.

Other Exercises

A. Lookup *roots of unity* on the internet, and note particularly facts about their summations and orthogonality. List three facts you learned that you hadn't known before about them.

B. Let $\zeta = e^{i\pi/n}$ be one of the nth primitive roots of unity in $\mathbb{C}$. Show that if $y \neq 0 \pmod{n}$, then

$$\sum_{w \in \mathbb{Z}_n} \zeta^{yw} = 0.$$

What happens if $y = 0 \pmod{n}$?

C. Read the Wikipedia page for Perron-Frobenius Theorem and answer the following questions.

(a) What is the difference between the cases of *positive* matrices and *nonnegative matrices*? What fails to work well, and how is it fixed? Which one applies in our situation or probability transition matrices?

(b) What does the simplicity of the Perron-Frobenius eigenvalue tell you about the corresponding eigenvector?

(c) Can a matrix for which Perron-Frobenius is applicable have more than one eigenvector with all positive components? Why is this important for Exercise 3.1 in the text?

(d) What is the P-F eigenvalue and corresponding eigenvector for the adjacency matrix $A(G)$ of a connected graph G?

(e) What is the P-F eigenvalue and corresponding eigenvector for the probability transition matrix $M(G)$ of a connected graph G?

(f) Use Sage to check the Perron-Frobenius theorem on two matrices of order $n \geq 5$ for which the hypotheses hold. Next find examples (again with $n \geq 5$) for which its conclusions fail even though in one case the matrix has all nonnegative entries and in another the matrix is irreducible.

(a) Suppose that two matrices M and N have a common eigen**vector** $\mathbf{v}$, with corresponding eigenvalues μ and ν respectively. Show that $\mathbf{v}$ is also an eigenvector for $M + N$.

(b) Suppose that two $p \times p$ matrices M and N have a complete set of linearly independent eigenvectors in common. Prove that M and N commute with each other, i.e., $MN = NM$.

(c) Suppose that two $p \times p$ matrices M and N have a complete set of linearly independent eigenvectors in common. How does $\operatorname{spec} M + N$ relate to $\operatorname{spec} M$ and $\operatorname{spec} N$?