Math 297Q/Roby **Midterm** 30 October 2006

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. You are asked to distribute 6 balls into 3 boxes. Compute the number of ways to do this for each of the twelve cases below.

Balls	Boxes	Arbitrarily	With at most 1 ball/box	With at least 1 ball/box
dist.	dist.			
dist.	indist.			
indist.	dist.			
indist.	indist.			

2. Where the order of purchasing is considered, let a_n be the number of ways to spend n dollars if for \$1 we can buy either a red ball or a white ball, and for \$2 we can buy either a blue, green, or black ball.
   (a) Show that a_n satisfies the recursion $a_n = 2a_{n-1} + 3a_{n-2}$ for $n > 1$, with $a_0 = 1$ and $a_1 = 2$.
   (b) Show that the ordinary generating function for a_n is $\dfrac{1}{1-2x-3x^2}$.
   (c) Decompose the above generating function into partial fractions to derive an explicit formula for a_n.

3. How many compositions of 19 use only the parts 2 and 3?

4. How many functions $f : [5] \to [5]$ are at most two-to-one (i.e., no more than two elts of [5] map to the same elt under f)?

5. Prove the following identity:
$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0}.$$

6. Let $S_l(n,k)$ denote the number of partitions of $[n]$ into k blocks, each of which contains *at least* ℓ elements. Show that
$$S_\ell(n+1,k) = \binom{n}{\ell-1} S_\ell(n-\ell+1, k-1) + kS_\ell(n,k)\,.$$

7. Prove that every set of ten (distinct) numbers from [100] contains two distinct nonempty subsets with the same sum. For extra credit, replace “distinct” with “disjoint”.

8. Prove that if n is odd, then a partition of n whose third part is 2 cannot be self-conjugate.