Math 2710 (Roby) **Practice Midterm #2** Spring 2013

1. Give a careful proof by induction that for every positive integer n

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

2. **PODASIP:** For any odd positive integer m, the number of nonzero perfect squares in $\mathbb{Z}_m$ is $\frac{m-1}{2}$.

3. **PODASIP:** For any $a \in \mathbb{Z}$ and any positive prime p, we have

$$a^{p-1} \equiv 1 \pmod{p}$$

4. (a) State carefully the *definition* of $\varphi(m)$, where m is a positive integer. (Do not give a formula for computing it.)

   (b) Working directly from this definition proof that

$$\varphi(m) = m - 1 \iff m \text{ is prime.}$$

5. Numerical & Computational problems

   (a) Expand $\left(x + \dfrac{1}{x}\right)^6$.

   (b) Compute $2^{327} \pmod{51}$.

   (c) Simplify $\dfrac{2\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + \sqrt{2}}$.

   (d) Compute the following (without a calculator!):

$$9^7 + 7 \cdot 9^6 + 21 \cdot 9^5 + 35 \cdot 9^4 + 35 \cdot 9^3 + 21 \cdot 9^2 + 7 \cdot 9$$

6. **True/False & Explain:** For each statement below, state whether it is true or false and give a convincing reason.

   (a) $\sqrt{3} + \sqrt{27} - \sqrt{48}$ is irrational.

   (b) The sum of a rational number and an irrational number is irrational.

   (c) For $0 \leq k \leq n$ we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

   (d) If $x \equiv a \pmod{m}$ and $x \equiv a \pmod{n}$, then $x \equiv a \pmod{mn}$.

7. Make sure you know how to prove the following facts from the text.

(a) Every integer $n > 1$ can be written as a product of primes (not necessarily uniquely). [Via strong induction.]

(b) For $1 \leq r \leq n$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

(c) (Euler-Fermat) If m is a positive integer and $(a, m) = 1$, then

$$a^{\varphi(m) \equiv 1 \pmod{m}}.$$

(d) There are numbers which are not rational.