1. Give a careful proof by induction that for every positive integer $n$

\[1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}\]

2. **PODASIP:** For any odd positive integer $m$, the number of nonzero perfect squares in $\mathbb{Z}_m$ is $\frac{m-1}{2}$.

3. **PODASIP:** For any $a \in \mathbb{Z}$ and any positive prime $p$, we have

\[a^{p-1} \equiv 1 \pmod{p}\]

4. (a) State carefully the definition of $\varphi(m)$, where $m$ is a positive integer. (Do not give a formula for computing it.)

(b) Working directly from this definition proof that

\[\varphi(m) = m - 1 \iff m \text{ is prime.}\]

5. Numerical & Computational problems

(a) Expand \(\left(x + \frac{1}{x}\right)^6\).

(b) Compute \(2^{327} \pmod{51}\).

(c) Simplify \(\frac{2\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + \sqrt{2}}\).

(d) Compute the following (without a calculator!):

\[9^7 + 7 \cdot 9^6 + 21 \cdot 9^5 + 35 \cdot 9^4 + 35 \cdot 9^3 + 21 \cdot 9^2 + 7 \cdot 9\]

6. **True/False & Explain:** For each statement below, state whether it is true or false and give a convincing reason.

(a) $\sqrt{3} + \sqrt{27} - \sqrt{48}$ is irrational.

(b) The sum of a rational number and an irrational number is irrational.

(c) For $0 \leq k \leq n$ we have

\[\binom{n}{k} = \binom{n}{n-k}.

(d) If $x \equiv a \pmod{m}$ and $x \equiv a \pmod{n}$, then $x \equiv a \pmod{mn}$.  

7. Make sure you know how to prove the following facts from the text.

(a) Every integer $n > 1$ can be written as a product of primes (not necessarily uniquely). [Via strong induction.]

(b) For $1 \leq r \leq n$

\[
\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}
\]

(c) (Euler-Fermat) If $m$ is a positive integer and $(a, m) = 1$, then

\[a^{\varphi(m)} \equiv 1 \pmod{m}.
\]

(d) There are numbers which are not rational.