Math 2710 (Roby) **Practice Final** Solutions Spring 2013

This is a closed book, closed note exam, except that you may have one **4** $\times$ **6** inch notecard with anything you like written on it front and back. You may use a calculator. Please do not discuss this exam with anyone other than the proctor during the exam.

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. Go over all the midterm, practice midterm, and midterm rewrite problems. Make sure you know how to do all of these, and understand any places where you lost points.

2. Go over your old homework problems. Make sure you know how to do all of these, and understand any places where you lost points or needed to rewrite.

3. Make sure you can state and give careful proofs of the following important theorems.

   (a) *Euclid's Lemma:* For any $a, b \in \mathbb{Z}$ and any prime p: $p \mid ab \implies p \mid a$ or $p \mid b$. (You may state without proof that the Euclidean algorithm gives solutions to certain linear diophantine equations.)

   (b) *Fermat's little theorem* and its generalization by Euler.

   (c) *Fundamental Theorem of Arithmetic*, i.e., unique factorization into primes in $\mathbb{Z}$.

   (d) *The Binomial Theorem.*

   (e) The rational numbers are countable, but the real numbers are uncountable.

   (f) The limit of the sum of two sequences is the sum of the limits.

   (g) If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

4. Prove or Disprove and Salvage if Possible:

   (a) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is onto.

   FALSE: Since for any $r \in \mathbb{R}$, $r^2 \geq 0$, there is no $x \in \mathbb{R}$ such that $x^2 = -1$, so f is not onto.

   SALVAGE: True for $f : \mathbb{R}^+ \to \mathbb{R}^+$.

   (b) Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are bounded iff fg is bounded.

   FALSE: Take $f(x) = x$ and $g(x) = 0$. Then $fg(x) = 0$ for every x, which is bounded even though f is clearly unbounded.

   SALVAGE: The direction ($\implies$) is true. For if $|f| \leq M$ and $|g| \leq N$ for some positive bounds M, N, then $|fg| = |f||g| \leq MN$, so fg is bounded.

   (c) If two functions $f, g : \mathbb{R} \to \mathbb{R}$ are monotone then $g \circ f$ is monotone.

   TRUE. Just use the definition.

(d) Suppose that a real number r satisfies $r \leq M + \varepsilon$ for all $\varepsilon > 0$. Then $r \leq M$.

TRUE. Suppose by way of contradiction that $r > M$. Set $\varepsilon = \frac{r-M}{2}$. Since the average of two numbers lies between them, we get

$$b > \frac{b+L}{2} = L + \frac{(b-L)}{2} = b + \varepsilon\,,$$

which is a contradiction. Hence, $r \leq M$.

(e) If a sequence $\{x_n\}$ of real numbers converges, then there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < 1/2^n$.

FALSE. Let $x_n = \sum_{k=0}^{n-1} (\frac{2}{3})^k$. This is the sequence of partial sums of a geometric series which converges to $\dfrac{1}{1-2/3} = 3$. But $x_{n+1} - x_n = (2/3)^n > (1/2)^n$.

(f) The sequence $\{x_n\}$ defined by $x_1 = 1$ and for every $n \in \mathbb{N}$

$$x_{n+1} = \frac{1}{x_1 + \cdots + x_n}$$

converges. (No need to find the limit if it exists.)

TRUE. First note that $x_1 = 1 > 0$, and that the positive real numbers are closed under addition and taking reciprocals. So if $x_i > 0$ for every $i \in [k]$, then $x_{k+1} > 0$ also. Hence, by induction, $x_n > 0$ for all $n \in \mathbb{N}^+$.

Next, since $x_n > 0$, we get $\displaystyle\sum_{i=1}^{n} x_i > \sum_{i=1}^{n-1} > 0 \implies x_{n+1} < x_n$. So the sequence $\{x_n\}$ is monotone and bounded (below by 0, above by 1); hence by MTC it converges.

(g) If $a_n \to 0$ and $b_n \to 0$, then $\sum a_n b_n$ converges.

FALSE. Let $a_n = b_n = \dfrac{1}{\sqrt{n}}$. Then $a_n \to 0$ and $b_n \to 0$, but $\sum a_n b_n = \sum \frac{1}{n}$, which is the harmonic series, which diverges.

(h) The sequence $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots$ converges.

FALSE. If $\sum \frac{1}{2n-1}$ converged, then so would $\sum \frac{2}{2n-1}$. But $\frac{1}{n} < \frac{2}{2n-1}$, so $\sum \frac{2}{2n-1}$ diverges by the comparison test.

5. Prove that for any sets A and B, $(A \cup B) \cap A^c = B - A$.

Let $x \in (A \cup B) \cap A^c$. Then $x \in (A \cup B)$ AND $x \in A^c$, so x is in A OR B, AND $x \notin A$. The last condition means that $x \in B$, since otherwise x could not be in $A \cup B$. Thus, $x \in B$, but $x \notin A$, so $x \in B - A$. This shows that $(A \cup B) \cap A^c \subseteq B - A$.

Conversely, let $x \in B - A$. Then $x \in B$ and $x \notin A$, which imply that $x \in A \cap B$ AND $x \in A^c$. Hence, $x \in (A \cup B) \cap A^c$. This shows that $(A \cup B) \cap A^c \supseteq B - A$. Combined with the first containment, we get the statement to be proven.

6. Let $\{a_n\}$ be a sequence with $a_1 = 1$ and $a_{n+1} = a_n + 3n(n+1)$ for $n \in \mathbb{N}$. Prove that $a_n = n^3 - n + 1$ for all $n \in \mathbb{N}$.

   We use mathematical induction. For $n = 1$, we have $a_1 = 1$, and the formula yields $1^3 - 1 + 1 = 1$, so the statement holds. Now suppose that the statement is true for some $k \in \mathbb{N}$, i.e., that $a_k = k^3 - k + 1$. Then the recurrence gives

$$\begin{aligned} a_{k+1} &= a_k + 3k(k+1) \\ &= k^3 - k + 1 + 3k(k+1) \\ &= k^3 + 3k^2 + 2k + 1 \\ &= (k^3 + 3k^2 + 3k + 1) - (k+1) + 1 \\ &= (k+1)^3 - (k+1) + 1 \end{aligned}$$

   where the second equality uses the induction hypothesis. So the statement also holds for $n = k+1$ if it holds for $n = k$. Hence, by the Principle of Mathematical Induction, the statement holds for every $n \in \mathbb{N}$.

7. Give the negation of the following statements, avoiding locutions like "It is not the case that…".

   (a) Fred goes bowling and says "Yabba-Dabba-Doo!"

   Either Fred doesn't go bowling or fails to say "Yabba-Dabba-Doo".

   (b) If I take the test, I'll fail.

   I take the test and don't fail.

   (c) Every action has an equal and opposite reaction.

   There is *some* action whose reaction is either unequal or in the same direction.

   (d) The product of any two odd numbers is prime.

   There exists a pair of odd numbers whose product is composite.

8. Let $f : A \to B$ and $g : B \to C$. Set $h = g \circ f$. Prove or Disprove each statement:

   (a) If h is injective, then f is injective.

   TRUE. If f is not injective, then there exists distinct elements $x, y \in A$ such that $f(x) = f(y)$. Since g is a function, this forces $g(f(x)) = g(f(y)) \iff h(x) = h(y)$. Then h is not injective. Thus, we have shown the contrapositive of the given statement.

   (b) If h is injective, then g is injective.

   FALSE. Let $A = \{1\}$, $b = \{a, b\}$, and $C = \{\alpha\}$. Define $f(1) = a$, $g(a) = g(b) = \alpha$.

   (c) If h is surjective, then f is surjective.

   FALSE. Let $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{\alpha\}$. Set $f(1) = f(2) = a$ and $g(a) = g(b) = \alpha$. Then $h(1) = h(2) = \alpha$, so h is surjective, but g is not.

(d) If h is surjective, then g is surjective.

TRUE. If h is surjective, then $C = h(A) = g(f(A))$. But since $f(A) \subseteq B$, we get $C = g(f(A)) \subseteq g(B)$, which by definition lies within C. Hence, $g(B) = C$ and g is surjective.

9. (a) Define carefully what it means for a sequence of real numbers to be a *Cauchy* sequence.

A sequence $\{a_n\}_{n\in\mathbb{N}}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N\,.$$

(b) Prove that any sequence of real numbers that converges must be a Cauchy sequence.

See the proof of Prop. 14.13.

10. A runaway train is hurtling towards a brick wall at the speed of 100 miles per hour. When it is two miles from the wall, a fly begins to fly repeatedly between the trains and the wall at the speed of 200 miles per hour. Determine how far the fly travels before it is smashed.

(1) If the fly is at the train when it is x miles from the wall, and y is the distance from the wall when the fly is next at the train, then the fly travels $x + y$ in the time t it takes the train to go $x - y$ miles. Thus, $\dfrac{x+y}{200} = t = \dfrac{x-y}{100} \implies y = x/3$, and the fly has traveled $x + y = 4x/3$. In the next segment, the fly travels $(4/3)(x/3)$, so the total distance the fly travels is given by

$$(4/3)x \sum_{n=0}^{\infty} (1/3)^n = \frac{4x}{3} \frac{1}{1 - 1/3} = 2x\,.$$

This gives the answer 4 miles when $x = 2$.

(2) The fly travels twice as fast as the train during a period in which the train covers two miles; hence, the fly must travel four miles.