Math 227Q/Roby **Practice Midterm 2** October 2005

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam no calculators are to be used.

1. Let A be the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix}$

   (a) Compute $\det A$.

   (b) Without finding A^{-1}, find $\det(A^{-1})$.

   (c) Use Cramer's Rule to find x_2 so that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}$.

2. Find the volume of the parallelepiped determined by the vectors, $\begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ -1 & -2 & 0 & 1 & 3 \\ 2 & 4 & 4 & 2 & 2 \end{bmatrix}$. Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$. What should the sum of the dimensions of these two subspaces be? Does your answer check?

4. Define a transformation $T : \mathbb{P}_3 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Describe the kernel and range of this linear transformation.

   (c) Write the matrix A of this linear transformation in terms of the standard bases for $\mathbb{P}_3$ and $\mathbb{R}_2$.

   (d) Compute a basis for $\operatorname{Nul} A$.

   (e) Compute a basis for $\operatorname{Col} A$.

5. Find the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ for the matrix $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

6. Prove or Disprove and Salvage if possible:
   (a) If A is a 2×2 matrix with a zero determinant, then one column of A is multiple of the other.
   (b) If A and B are $n \times n$ matrices with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$.
   (c) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$.
   (d) $\det A^T = -\det A$.

7. Decide whether each statement below is True of False. Justify your answer.
   (a) The number of pivot columns of a matrix equals the dimension of its column space.
   (b) Any plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.
   (c) The dimension of the vector space $\mathbb{P}^4$ is 4.
   (d) If $\dim V = n$ and S is a linearly indpendent set in V, then S is a basis for V.
   (e) if there exists a linearly dependent set $\{v_1, \ldots, , v_p\}$ that spans V, then $\dim V \leq p$.

8. Prove or Disprove and Salvage if possible:
   (a) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$
   (b) The range of a linear transformation is a vector subpace of the codomain.
   (c) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$.
   (d) The row space of A^T is the same as the column space of A.

9. If A is a 4×3 matrix, whas is the largest possible dimension of the row space of A? What is the smallest possible dimension? What if A is 3×4 matrix? Explain!

10. Let $\mathcal{S} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$.
    (a) Is $\mathcal{S}$ linearly independent in $\mathbb{P}_2$? Explain!
    (b) Is $\mathcal{S}$ a basis for $\mathbb{P}_2$? Explain!
    (c) Express the polynomial $\mathbf{p}(t) = 3 + t - 6t^2$ as a linear combination of elements of $\mathcal{S}$.
    (d) Is the expression unique? Explain!

11. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain!

12. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.