CHANGE OF BASIS

§4.7 Supplemental Lecture Notes

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<See book example 1>

Summary:

B basis vectors written with C coordinates. (i.e. $[b_1]_{C}$, $[b_2]_{C}$)

Any vector x in B-coord.

Same vector x rewritten in C-coord.

$$\underbrace{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}}_{\underset{C\leftarrow B}{P}} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{x_B} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{x_C}$$

Thinking questions

① In the example above, what is x in standard $\mathbb{R}^2$ coord?

② How do you write the B basis vectors in B coord?

③ How do you write the C basis vectors in C coord?

④ How do you write the B basis vectors in C coord?

⑤ How do you write the C basis vectors in B coord?

⑥ How do you know $\underset{C\leftarrow B}{P}$ is invertible?

To check

- Interpret what it means to plug in $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ instead of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$:

$$\underbrace{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}}_{\underset{C\leftarrow B}{P}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_C$$

does it make sense?

- Find $\underset{B\leftarrow C}{P}$ and apply it to $\begin{bmatrix} 6 \\ 4 \end{bmatrix}_C$. What do you expect to get?

$$\underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{P} \begin{bmatrix} 6 \\ 4 \end{bmatrix}_C = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$\boxed{\S 4.7}$ cont. What if we know both $\mathcal{B}$ and $\mathcal{C}$ in terms of standard $\mathbb{R}^2$-coord?

<see book example 2>

Given: $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} -9 \\ 1 \end{bmatrix}}_{b_1}, \underbrace{\begin{bmatrix} -5 \\ -1 \end{bmatrix}}_{b_2} \right\}$ and $\mathcal{C} = \left\{ \underbrace{\begin{bmatrix} 1 \\ -4 \end{bmatrix}}_{c_1}, \underbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{c_2} \right\}$

Want to find: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, i.e. how to write $\mathcal{B}$ in terms of $\mathcal{C}$

Method: Solve $\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$

Then $\underbrace{\begin{bmatrix} 1 \\ -4 \end{bmatrix}}_{c_1} x_1 + \underbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{c_2} x_2 = \underbrace{\begin{bmatrix} -9 \\ 1 \end{bmatrix}}_{b_1}$ so we will have $b_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{\mathcal{C}}$

And $\underbrace{\begin{bmatrix} 1 \\ -4 \end{bmatrix}}_{c_1} y_1 + \underbrace{\begin{bmatrix} 3 \\ -5 \end{bmatrix}}_{c_2} y_2 = \underbrace{\begin{bmatrix} -5 \\ -1 \end{bmatrix}}_{b_2}$ so we will have $b_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{\mathcal{C}}$

Solution $\left[c_1 \;\; c_2 \;\vdots\; b_1 \;\; b_2 \right] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$

$\sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$

So $[b_1]_{\mathcal{C}} = \begin{bmatrix} \quad \end{bmatrix}$, $[b_2]_{\mathcal{C}} = \begin{bmatrix} \quad \end{bmatrix}$ and $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} \qquad \end{bmatrix}$

Summary

$\left[\mathcal{C} \;\vdots\; \mathcal{B} \right]$

$\downarrow \sim$

$\left[I \;\vdots\; \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} \right]$