Math 2210Q (Roby) **Practice Midterm #2** **Solutions** Spring 2014

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

For this exam no calculators are to be used.

1. Let A be the matrix $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & -2 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

   (a) Compute $\det A$. The answer is 6. This may be computed in a couple of ways: (1) by doing row reductions to transform A to triangular form, keeping track of any moves that modify the determinant or (2) expanding by cofactors (minors) along a suitable row or column.

   (b) Compute $\det(A^{-1})$ without computing A^{-1}. Since $\det(A^{-1}) = 1/(\det A)$, the answer is $1/6$.

   (c) Use Cramer's Rule to find x_4 so that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

   To apply Cramer's rule, for x_4, we replace the fourth column of A with the output vector, take the determinant, and divide that by the determinant of the original matrix (computed above to be 6). Therefore,

$$x_4 = \frac{1}{6} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 2 & -2 & 1 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{6}\left((-1) \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & -2 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & 2 \\ 1 & 0 & 0 \end{vmatrix} \right) \quad = \frac{1}{6}(-12 - 6) = -3$$

2. Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix}3\\6\\7\end{bmatrix}$, $\begin{bmatrix}0\\4\\1\end{bmatrix}$, and $\begin{bmatrix}2\\3\\4\end{bmatrix}$.

Take the absolute value of the determinant of the matrix of column vectors to get:

$$\left\| \begin{matrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ 7 & 1 & 4 \end{matrix} \right\| = |-5| = 5.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ -1 & -2 & 0 & 1 & 3 \\ 2 & 4 & 4 & 2 & 2 \end{bmatrix}$. Find bases for Col A and Nul A. What should the sum of the dimensions of these two subspaces be? Does your answer check?

By row reduction we see that $A \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has pivots in columns 1 and 3, so we use those columns of A as a basis for Col A. For Nul A, we parameterize the solutions in terms of the free variables to get the basis shown below.

$$\text{Col}\,A = \text{Span}\left\{ \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix} \right\}, \text{ and } \text{Nul}\,A = \text{Span}\left\{ \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\-2\\0\\1\end{bmatrix} \right\}$$

4. Define a transformation $T : \mathbb{P}_3 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix}\mathbf{p}(0)\\\mathbf{p}(2)\end{bmatrix}$.

(a) Show that T is a linear transformation.

Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}_3$. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix}(\mathbf{p}+\mathbf{q})(0)\\(\mathbf{p}+\mathbf{q})(2)\end{bmatrix} = \begin{bmatrix}\mathbf{p}(0)+\mathbf{q}(0)\\\mathbf{p}(2)+\mathbf{q}(2)\end{bmatrix} = \begin{bmatrix}\mathbf{p}(0)\\\mathbf{p}(2)\end{bmatrix} + \begin{bmatrix}\mathbf{q}(0)\\\mathbf{q}(2)\end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}).$$

Similarly, one shows that $T(c\mathbf{p}) = cT(\mathbf{p})$ for any $c \in \mathbb{R}$.

(b) Describe the kernel and range of this linear transformation.

By def. $\ker T$ is the set of polys that map to $\begin{bmatrix}0\\0\end{bmatrix}$, i.e., $\{\mathbf{p} \in \mathbb{P}_3 : \mathbf{p}(0) = \mathbf{p}(2) = 0\}$, while range T is all of $\mathbb{R}^2$, since for any $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$, we can always find a polynomial $\mathbf{p}$ with $\mathbf{p}(0) = a$ and $\mathbf{p} = b$ (e.g., by Lagrange interpolation, or less fancily by noting that matrix below has two pivot columns, so the dimension of range $T = \dim \text{Col}\,A = 2$).

(c) Write the matrix A of this linear transformation in terms of the standard bases for $\mathbb{P}_3$ and $\mathbb{R}_2$.

(d) Compute a basis for $\operatorname{Nul} A$.

(e) Compute a basis for $\operatorname{Col} A$. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}$, which is already almost in RREF. So we get bases $\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$.

5. Find the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ for the matrix $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

We have $\dim \operatorname{Col} A = \operatorname{rk} A = \#\text{pivot cols} = 3$, and $\dim \operatorname{Nul} A = \#\text{free vars} = 6-3 = 3$, by the rank-nullity theorem.

6. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? What is the smallest possible dimension? What if A is 3×4 matrix? Explain!

Since $\operatorname{Row} A$ is spanned by 4 vectors in $\mathbb{R}^3$, it has dimension at most 3, and $A = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}$, shows that dimension is achievable. The smallest possible dimension is 0, achieved by $A = 0$. Similar reasoning shows the same bounds for a 3×4 matrix.

7. Prove or Disprove and Salvage if possible:

(a) If A is a 2×2 matrix with a zero determinant, then one column of A is multiple of the other. T

(b) If λ is an eigenvalue of an $n \times n$ matrix M, then λ^2 is an eigenvalue of M^2. T

(c) If A and B are $n \times n$ matrices with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$. F

(d) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$. T

(e) $\det A^T = -\det A$. F

8. Decide whether each statement below is True of False. Justify your answer.

(a) The number of pivot columns of a matrix equals the dimension of its column space. T

(b) Any plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. F

(c) The dimension of the vector space $\mathbb{P}^4$ is 4. F

(d) If $\dim V = n$ and S is a linearly indpendent set in V, then S is a basis for V. F

(e) If there exists a linearly dependent set $\{v_1, \dots, , v_p\}$ that spans V, then $\dim V \le p$. T

(f) The eigenvectors of any $n \times n$ matrix are linearly independent in $\mathbb{R}^n$. F

9. Prove or Disprove and Salvage if possible:

(a) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$ F

(b) The range of a linear transformation is a vector subpace of the codomain. T

(c) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$. T

(d) The row space of A^T is the same as the column space of A. T

10. Let $\mathcal{S} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$.

(a) Is $\mathcal{S}$ linearly independent in $\mathbb{P}_2$? Explain!

(b) Is $\mathcal{S}$ a basis for $\mathbb{P}_2$? Explain!

(c) Express $\mathbf{p}(t) = 3 + t - 6t^2$ as a linear combination of elements of $\mathcal{S}$.

(d) Is the expression unique? Explain!

The standard isomorphism $\mathbb{P}_2 \to \mathbb{R}^3$ given by $1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $t \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $t^2 \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ takes the polynomials in S to the columns of the matrix

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

which is invertible. So the columns of C form a basis for $\mathbb{R}^3$, which means the original set $\mathcal{S}$ is a basis for $\mathbb{P}_2$ (b). In particular, this means that there is a unique way of writing any polynomial as a linear combination of the basis elements (d). The usual techniques for solving $C\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ give $\mathbf{x} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$, so

$$\mathbf{p}(t) = 3 + t - 6t^2 = 7(1 - t^2) - 3(t - t^2) - 2(2 - 2t + t^2)\,.$$

as one can easily check.

11. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain! No. This system is equivalent to $A\mathbf{x} = b$, where $\dim \operatorname{Nul} A = 2 \implies \dim \operatorname{Col} A = 8 - 2 = 6 \implies \mathbf{x} \mapsto A\mathbf{x}$ is *onto* $\mathbb{R}^6$; hence, every right hand side is obtainable.

12. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points. Check!