Math 2210Q (Roby) **Practice Midterm #1 Solutions** Spring 2014

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators are to be used.**

1. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.

2. Compute all solutions to the following linear system by reducing its associated augmented matrix to Reduced Row-Echelon Form.

$$\begin{array}{rcrcrcrl} x_1 & + & & & x_3 & & & = 2 \\ x_1 & + & 2x_2 & + & 5x_3 & + & 2x_4 & = 0 \\ 2x_1 & + & x_2 & + & 4x_3 & & & = 2 \\ x_1 & + & x_2 & + & 3x_3 & + & x_4 & = 1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 1 & 2 & 5 & 2 & 0 \\ 2 & 1 & 4 & 0 & 2 \\ 1 & 1 & 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|r} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ -2 - 2x_3 \\ x_3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$. Write each of the following vectors as a linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, or show that this cannot be done.

   (a) $\begin{bmatrix} -2 \\ 14 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$

   (a) Row reduce as follows:

$$\left[\begin{array}{rr|r} 1 & 2 & -2 \\ 2 & -2 & 14 \\ 3 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{bmatrix} -2 \\ 14 \\ 9 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 3 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

   (b)

$$\left[\begin{array}{rr|r} 1 & 2 & 10 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & 10 \\ 0 & 1 & 8/3 \\ 0 & 1 & 29/5 \end{array}\right]$$

   which is inconsistent, so no solution.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - 3y \\ x + xy \end{bmatrix}$. Is T a **linear** transformation or not? Give a complete and careful explanation.

We compute $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-2\\2\end{bmatrix}$ while $T\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}6\\-4\\6\end{bmatrix} \neq 2 \cdot T\begin{bmatrix}1\\1\end{bmatrix}$, so T is not linear.

5. Perform the indicated matrix operations, where $A = \begin{bmatrix}-1 & 2\\1 & 3\end{bmatrix}$, $B = \begin{bmatrix}-1 & 0 & 4\\2 & 1 & 0\end{bmatrix}$, $C = [2 \quad 4 \quad -1]$, and $D = [-2 \quad -3]$. If any operation is not possible, explain why.

   a) A^2 $\begin{bmatrix}3 & 4\\2 & 11\end{bmatrix}$ b) $B^T A$ $\begin{bmatrix}3 & 4\\1 & 3\\-4 & 8\end{bmatrix}$ c) DAB $[-25 \quad -13 \quad -4]$

   d) BC Incompatible dimensions: 2×3 and 1×3. e) D^3 Ditto: 1×3 and 1×3.

6. Let R be the linear transformation on two-dimensional real vectors that multiplies the vector components by 2. Let S be the linear tranformation which projects vectors onto the first coordinate (i.e., the x-axis). Let T be the linear transformation which rotates a vector by $\pi/2$ radians.

   a) Write a matrix for the composite linear tranformation which performs R, then S, then T (in that order)

   For each transformation, compute its action on the standard basis $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$, and write the images as the columns of the matrix. This leads to

$$R = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \quad S = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad \text{and } T = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \implies TSR = \begin{bmatrix}0 & 0\\2 & 0\end{bmatrix}.$$

   b) use your answer to the prevous part to compute the image of the vector $\mathbf{u} = \begin{bmatrix}-3\\4\end{bmatrix}$ under this composite tranformation.

   One easily computes that $TSR\mathbf{u} = \begin{bmatrix}0\\-6\end{bmatrix}$. Or one can apply each transformation in turn (just based on its geometrical description, if you like) to $\mathbf{u}$, getting $R\mathbf{u} = \begin{bmatrix}-6\\8\end{bmatrix}$, $SR\mathbf{u} = \begin{bmatrix}-6\\0\end{bmatrix}$, etc.

   c) Is the composite tranformation injective (one-to-one)? You may give your answer in clear English if you feel that no calculation is necessary.

   No. One convincing argument is to note that if $\mathbf{v} = \begin{bmatrix}-3\\1\end{bmatrix}$, then $TSR\mathbf{v} = \begin{bmatrix}-6\\0\end{bmatrix} = TSR\mathbf{u}$. Or you could state that since the transformation S always zeroes out the second coordinate of any vector, that the image of any vector only depends on its first coordinate.

7. Prove or Disprove and Salvage if possible:

   (a) If A and B are matrices such that both products AB and BA are defined, then A and B must be square matrices of the same size.

   False. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.
   Then AB and BA are both defined, since either way the number of columns in the first factor equals the number of rows in the second. (Note that the entries are irrelevant here; only the dimensions matter.)

   (b) If A, B, and C are matrices such that $AB = AC$, then $B = C$.

   False. Let $A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$. Then $AB = AC = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, but $B \neq C$.

   (c) If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

   True. See proof in text.

   (d) If two vectors in $\mathbb{R}^n$ are linearly dependent, then they both lie on the same line through the origin.

   True. Let $c\mathbf{v} + d\mathbf{w} = 0$, where $\mathbf{v}, \mathbf{w}$ are vectors and $c, d \in \mathbb{R}$ are not both zero; WLOG (without loss of generality) say $d \neq 0$. Then we can write $\mathbf{w} = \frac{-c}{d}\mathbf{v}$, so $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$, and must lie on the line through $\mathbf{v}$ and the origin. (In the special case that $\mathbf{v} = 0$, then $\mathbf{w} = 0$ as well, and the statement clearly holds.)

   (e) If A is **any** $k \times k$ matrix, then the block matrix $\begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ is invertible. True. By straightfoward matrix calculation we have:

   $$\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} = \begin{bmatrix} I+0 & 0+0 \\ A+(-A) & 0+I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I_{2k}$$

   So by IMT, the matrix $\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}$ is a (two-sided) inverse.

8. Compute the inverse of each of the following matrices (if they exist). Show your work. (If you use a special formula, state the formula before applying it.) If an inverse does not exist, explain clearly why.

   a) $\begin{bmatrix} 1 & 6 \\ -1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 4 & 1 & 6 \\ 1 & 2 & 1 & 2 \\ -1 & -1 & -1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \end{bmatrix}$

   (a) By the formula for inverting a 2×2 matrix, we get

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{2-(-6)} \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & -3/4 \\ 1/8 & 1/8 \end{bmatrix}.$$

b) $\begin{bmatrix} 0 & -1 & 3 & 2 \\ \frac{2}{3} & \frac{2}{3} & -\frac{7}{3} & -\frac{5}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{4}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{5}{3} & \frac{4}{3} \end{bmatrix}$

c) DNE.

9. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(u) = T(v)$ for some pair of distinct vectors u and v in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not? No!

10. Show that if a product of two $n \times n$ matrices AB is invertible, then so is A.

    If AB is invertible, then $\exists$ a matrix W s.t. $(AB)W = I \implies A(BW) = I$ by associativity. Hence (BW) is a right inverse for A, so by IMT, A is invertible.

    Note that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \implies AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so the statement is *false* if we remove the condition that A and B must be square.