1. Define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$.

(a) Find the image under $T$ of $p(t) = 5 + 3t$.
(b) Show that $T$ is a linear transformation.
(c) Find the matrix for $T$ relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.
(d) Is $T$ one-to-one? Is $T$ onto? Explain!

2. Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

3. Show that if $T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, then $\det T = (b - a)(c - a)(c - b)$.

4. Prove or (use a counterexample to) Disprove and Salvage if possible:
   (a) If $A = QR$, where $Q$ has orthonormal columns, then $R = Q^T A$.
   (b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then $S$ is linearly independent.
   (c) If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$.
   (d) Each eigenvector of a square matrix $A$ is also an eigenvector of $A^2$.
   (e) There exists a $2 \times 2$ matrix with real entries that has no eigenvectors in $\mathbb{R}^2$.
   (f) If $A$ is row equivalent to the identity matrix $I$, then $A$ is diagonalizable.

5. Decide whether each statement below is True of False. Justify your answer. For False statements, a counterexample is usually best. Extra credit for good salvages and more for proofs thereof!
   (a) If $y$ is in a subspace $W$, then the orthogonal projection of $y$ onto $W$ is $y$ itself.
   (b) For an $m \times n$ matrix $A$, vectors in $\text{Nul} A$ are orthogonal to vectors in $\text{Row} A$.
   (c) The matrices $A$ and $A^T$ have the same eigenvalues, counting multiplicities.
   (d) A nonzero vector can correspond to two different eigenvalues of $A$.
   (e) The sum of two eigenvectors of a square matrix $A$ is also an eigenvector of $A$. 
6. If a $n \times n$ matrix $A$ satisfies $A^2 = A$, what can you say about the determinant of $A$?

7. Assume that matrices $A$ and $B$ below are row equivalent:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Without calculations, list rank $A$ and dim Nul $A$. Then find bases for Col $A$, Row $A$, and Nul $A$.

8. Find the maximum value of $Q(x) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.)

9. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss!

10. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.

11. Here are some specific tasks (modified from a list of Prof. Leibowitz) I expect you to be able to perform with demonstrated understanding:

   (a) Given a matrix $A$, find the dimensions of and bases for Col $A$, Nul $A$, and Row $A$. Use the relations among rank, dim Nul $A$, and size of $A$ to understand properties of the associated linear transformation (one-to-one, onto, kernel, range).

   (b) Use row reduction to solve linear equations, show that a given column vector is in the span of a given set of vectors, or compute the inverse of a matrix.

   (c) Use row operations to reduce a matrix $A$ to triangular form in order to calculate det $A$. Use properties to compute the determinant of related matrices.

   (d) Diagonalize a given matrix and use the $A = PDP^{-1}$ factorization to calculate a power of $A$.

   (e) Orthogonally diagonalize a real symmetric matrix, possibly representing a quadratic form, and compute constrained extrema of the form.

   (f) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

   (g) Understand how to use the LU factorization and the singular value decomposition of an $m \times n$ matrix.

   (h) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.

   (i) Show that a certain set of vectors is a subspace of a given space, or are eigenvectors of a certain matrix.

   (j) Use various forms of the Invertible Matrix Theorem in context.