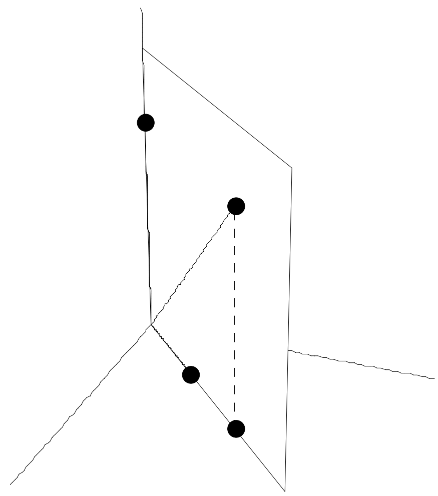
Section 6.4 The Gram-Schmidt Process

Goal: Form an orthogonal basis for a subspace W.

EXAMPLE: Suppose $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.



Let

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \hat{\mathbf{y}} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

(component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$)

EXAMPLE: Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution: Let

$$\mathbf{v}_1 = \mathbf{x}_1 \text{ and } \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for Span$\{\mathbf{x}_1, \mathbf{x}_2\}$.

Let

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

(component of $\mathbf{x}_3$ orthogonal to Span$\{\mathbf{x}_1, \mathbf{x}_2\}$)

Note that $\mathbf{v}_3$ is in W. Why?

$\Rightarrow$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W.

THEOREM 11 THE GRAM-SCHMIDT PROCESS

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a subspace W of $\mathbf{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W and

$$\text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_p\} = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$$

EXAMPLE Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, is a basis for a subspace W of $\mathbf{R}^4$. Describe an orthogonal basis for W.

Solution:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \text{ and}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix}$$

Replace $\mathbf{v}_2$ with $14\mathbf{v}_2$: $\mathbf{v}_2 = 14\begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$

(optional step - to make $\mathbf{v}_2$ easier to work with in the next step)

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{9}{630}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{70}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Rescale (optional): } \mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix}$$

Orthogonal Basis for W:

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix} \right\}$$

Orthonomal Basis

Suppose the following is an orthogonal basis for subspace $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$:

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$$

Rescale to form unit vectors:

$$\mathbf{u}_1 = \tfrac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \tfrac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \tfrac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormal basis for W : $\{\mathbf{u}_1, \mathbf{u}_2\}$

QR Factorization

THEOREM 12 (The QR Factorization)

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthogonal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its main diagonal.

EXAMPLE Find the QR factorization of $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$.

Solution: Use the Gram Schmidt process to find $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which is an orthonomal basis for

col A =span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$. So $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$.

Since U has orthonormal columns, $Q^T Q = I$. So if $A = QR$, then

$$\_\_\_\_A = \_\_\_\_QR$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3 \end{bmatrix}.$$