Math 215/Roby **Practice Midterm** October 2006

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam the use of calculators or determinants is strictly forbidden.

1. For each of the following statements, indicate whether the statement is true or false. If true, give a short explanation of why (full proofs unnecessary). If false, provide a counterexample. Try to "salvage" the false statements by providing a similar (non-trivial) statement that would be true.

   (a) Any union of subspaces of a vector space V is a subspace of V.

   (b) The set of all even functions $E = \{f : \mathbb{R} \mapsto \mathbb{R} \mid f(x) = -f(-x)\ \forall x \in \mathbb{R}\}$ is a subspace of $\{f : \mathbb{R} \mapsto \mathbb{R}\}$.

   (c) The vector space $\mathbb{R}^n$ (i.e., n-tuples of numbers) is a vector space over the complex numbers.

   (d) Let $V = \{(a, b) \mid a, b \in \mathbb{F}\}$, where $\mathbb{F}$ is any field. Define addition of elements of V coordinatewise, and scalar multiplication by

   $$c(a, b) = (a, b)$$

   Then V is a vector space over $\mathbb{F}$.

   (e) The polynomials $x^2 - 2$, $2x^2 + 3x$, and $-x^2 - 4x + 4$ form a basis for $P_2(\mathbb{R})$.

   (f) The dimension of the space of all $n \times n$ skew-symmetric matrices is $n(n + 1)/2$. (Recall a matrix is *skew-symmetric* if it equals the negative of its transpose.)

   (g) Subsets of linearly dependent sets are linearly dependent.

2. Let $\lambda \in \mathbb{C}$ be a root of $p \in \mathbb{R}[x]$. Show that $\bar{\lambda}$ is also a root of p.

3. State the definition of a basis for a vector space. Directly from this definition, show that a finite subset β of a vector space V is a basis for V if and only if every vector $v \in V$ can be expressed uniquely as a linear combination of vectors in β.

4. Suppose that V and W are both finite dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim W \leq \dim V$.

5. Does there exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals the following? Give an example or show that none can exist.

   $$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

6. Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbf{F} \setminus \{0\}$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

7. Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \ldots + x_n, \ldots, x_1 + \ldots + x_n);$$

in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

8. Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

9. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points. Compare them with the solutions handed out or done in class.