Kepler's Laws

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1 Introduction

Johannes Kepler was an astromoner in the late 16th and early 17th centuries. His most famous work is his three laws which describe the motion of the planets:

I The orbit of a planet traces out an ellipse in a fixed plane.

II Equal areas are swept out in equal times.

III Let T be the period of the orbit and a half the length of the major axis of the ellipse. Then $T = \tilde{c}a^{3/2}$, where $\tilde{c}$ is a constant independent of the planet. The constant $\tilde{c}$ may depend on the sun.

How did Kepler arrive at these laws? He examined centuries' worth of astronomical data, and came up with a model to fit it. Amazingly, Isaac Newton (probably around 1680) derived all three of

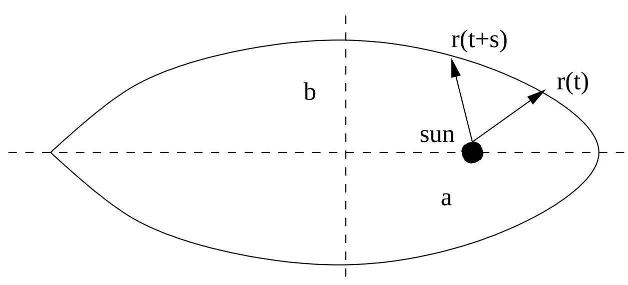


Figure 1: The bold-faced dot is the sun at one of the focii of the ellipse. The major and minor axes of the ellipse are a and b respectively.

Kepler's laws, starting with some reasonable physical assumptions. Let's make that precise. Newton showed that the only centripetal force which is consistent with Kepler's laws is an inverse square law:

$$|\mathbf{F}| = F = \frac{GmM}{r^2}. \tag{1}$$

Here $\mathbf{F}$ is the gravitational force between two object of mass M and m separated by a distance $r = |\mathbf{r}|$, and G is a universal constant. We can state his three laws of the planets as follows. He also used from the following three laws:

*These notes are mostly cribbed from lecture notes by Nick Korevaar, at the University of Utah

(a) A planet which orbits in a fixed plane containing the sun satisfies Kepler II if and only if the force is centripetal (i.e. $\mathbf{F}$ and $\mathbf{r}$ are parallel).

(b) The only centripetal acceleration consistent with Kepler I has magnitude c/r^2 (i.e. an inverse square law).

(c) Kepler III holds if and only if the constant c is (b) is the same for each planet.

We can reproduce Newton's arguments using the tools we've developed so far in class. Beyond the historical importance, it's facinating to see the ideas we've discussed in class used to derive equations which govern the fundamental behavior of our solar system.

2 Notation

We'll treat the planet and the sun as point particles, with their masses concentrated at their centers. Fixing our attention on one planet, we denote its position at time t by $\mathbf{r}(t)$. It's velocity and acceleration are, respectively

$$\mathbf{v}(t) = \mathbf{r}'(t) \qquad \mathbf{a}(t) = \mathbf{r}''(t).$$

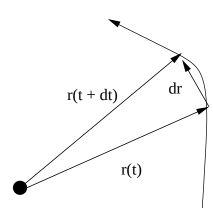
A force $\mathbf{F}$ is called *centripetal* if $\mathbf{F}$ and $\mathbf{r}$ are parallel; recall this is equivalent to $\mathbf{r} \times \mathbf{F} = \mathbf{0}$. As a final bit of notation, we will denote the magnitude of a vector by writing the same letter in regular typeface. For instance, $r = |\mathbf{r}|$ and $v = |\mathbf{v}|$.

3 Centripetal forces and equal areas

The following two lemmas show that Kepler II is equivalent to the force being a centripetal force. The first lemma

Lemma 1 *If the orbit of a planet lies in a plane containing the sun and Kepler II holds, then the acceleration is centripetal.*

Proof: Recall that the magnitude of the cross product of two vectors gives the area of the parallelogram they span, which implies



$$dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}| = \frac{1}{2}|\mathbf{r} \times \mathbf{r}' dt|.$$

(Why is this all true? The first equality is the geometric meaning of the cross product as an area and linear approximation. The second is the chain rule.) This gives us

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \mathbf{r}'|. \tag{2}$$

Now suppose the orbit of a planet lies in a plane with unit normal $\mathbf{n}$. Then $\mathbf{r}$ and *all its derivatives* lie in this plane. Now, Kepler II says $dA/dt = h$ for some constant h. Plugging this into equation (2), we see

$$\mathbf{r} \times \mathbf{r}' = 2h\mathbf{n}.$$

(We've also used that $\mathbf{r}$ and $\mathbf{r}'$ lie in a fixed plane, so their cross product has to be perpendicular to this plane.) Taking a derivative in time, we get

$$\mathbf{0} = \frac{d}{dt}(2h\mathbf{n}) = \frac{d}{dt}(\mathbf{r} \times \mathbf{r}')\mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r} \times \mathbf{r}''.$$

This last equation, $\mathbf{r} \times \mathbf{r}'' = 0$, says the acceleration is parallel to the position, i.e. the motion is centripetal. □

The next lemma proves the converse:

Lemma 2 *If the motion is centripetal, then the planet orbits in a fixed plane containing the sun Kepler II holds.*

Proof: Basically, we will run the argument of the last lemma in reverse. if $\mathbf{r}$ and $\mathbf{r}''$ are parallel, then

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}'') = \mathbf{r} \times \mathbf{r}'' = 0,$$

and so

$$\mathbf{r} \times \mathbf{r}' = \mathbf{N},$$

where $\mathbf{N}$ is a constant vector. In particular, $\mathbf{r} \cdot \mathbf{N} = 0$, so $\mathbf{r}$ lies in the plane through the origin which is perpendicular to $\mathbf{N}$. Now write $\mathbf{N} = 2h\mathbf{n}$, where $\mathbf{n}$ has length 1 and $h > 0$ is a constant. This is our familiar $\mathbf{r} \times \mathbf{r}' = 2h\mathbf{n}$, which then implies (using equation (2))

$$\frac{dA}{dt} = h.$$

□

4 Moving frames

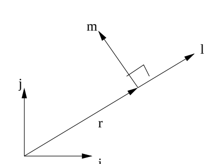
It will be convenient to use a new "moving" frame of reference, which is adapted to our moving planet. First, we may as well fix the plane of motion for the planet to be the $x - y$ plane, with $\mathbf{n} = \hat{k}$. Then

$$\mathbf{r}(t) = (x(t), y(t), 0) = (r\cos\theta, r\sin\theta, 0),$$

where we've used polar coordinates in the last equality. Remember that r and θ are functions of t! Define

$$\hat{l} = (\cos\theta, \sin\theta, 0) \qquad \hat{m} = (-\sin\theta, \cos\theta, 0).$$

Notice that we have



$$\mathbf{r}(t) = r(t)\hat{l}(t), \qquad \hat{l} \times \hat{m} = \hat{k}.$$

Lemma 3 *1.* $\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$*, and so* $\theta' = \frac{2h}{r^2}$

2. $\mathbf{r}' = r'\hat{l} + r\hat{l}'$

3. $\hat{l}' = \theta'\hat{m}$

4. $\hat{m}' = -\theta'\hat{l}$

Proof: In polar coordinates, the area element is given by $dA = (r^2/2)d\theta$, which gives $h = A' = (r^2/2)\theta'$. Rearranging this last equation, we get $\theta' = (2h)/r^2$, provong (1). The second item, $\mathbf{r}' = r'\hat{l} + r\hat{l}'$, follows from differentiating $\mathbf{r} = r\hat{l}$. The last two follow from the formulas for $\hat{l}$ and $\hat{m}$ and the chain rule. □

5 Elliptical orbits and Kepler III

First we show that if gravity is a centripetal force whose magnitude is $f(r) = c/r^2$, then the planets trace out ellipses and Kepler III holds. Then we show the converse: if all three of Kepler's laws hold then gravity must be a centripetal force whose magnitude is $f(r) = c/r^2$.

We start with a centripetal force, so

$$\mathbf{r}'' = -f\hat{l} = -\frac{f}{r}\mathbf{r}.$$

A priori (or, before we deduce anything), f can be a function of anything. Alright, now take some cross products:

$$\mathbf{r}\text{''} \times \hat{k} = -f\hat{l} \times \hat{k} = f\hat{m} = \frac{f}{\theta'}\hat{l}' = \frac{fr^2}{2h}\hat{l}'.$$

Suppose further that the force is an inverse square law: $f = f(r) = c/r^2$. Then

$$\mathbf{r}\text{''} \times \hat{k} = \frac{c}{r^2}\frac{r^2}{2h}\hat{l}' = \frac{c}{2h}\hat{l}'.$$

Integrate this last equation with respect to t and recall that $\hat{k}$ is constant:

$$\mathbf{r}' \times \hat{k} = \frac{c}{2h}\hat{l} + \mathbf{E},$$

where $\mathbf{E}$ is a constant of integration, which must be perpendicular to $\hat{k}$ (because everything else in the equation is). Now we have

$$\frac{c}{2h}\hat{l} + \mathbf{E} = \mathbf{r}' \times \hat{k} = (r'\hat{l} + r\hat{l}') \times k = (r'\hat{l} + r\theta'\hat{m}) \times k = -r'\hat{m} + r\theta'\hat{l}.$$

Take the dot product of both sides of the equation $c/(2h)\hat{l} + \mathbf{E} = -r'\hat{m} + r\theta'\hat{l}$ with $\hat{l}$:

$$\frac{c}{2h} = \mathbf{E} \cdot \hat{l} = r\theta' = \frac{2h}{r},$$

which we can rearrange to read

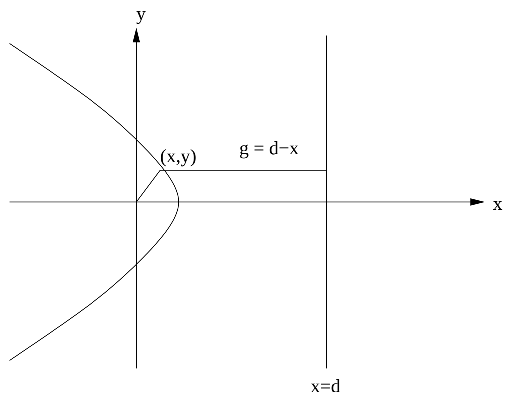
$$r = \frac{(4h^2/c)}{1 + \frac{2h}{c}\mathbf{E} \cdot (\cos\theta, sin\theta, 0)}.$$

Now, we may as well take $\mathbf{E} = (\lambda \cos\theta, 0, 0)$ (after tilting our heads by the correct angle), so we really have

$$r = \frac{4h^2/c}{1 + (2h\lambda/c)\cos\theta} = \frac{ed}{1 + e\cos\theta}, \tag{3}$$

where $e = (2h\lambda)/c$ and $d = (2h)/\lambda$. This is the polar equation for an ellipse with eccentricity e! (See the figure below.)

This is the general form of a conic section. Note that $g = d - x = d - r\cos\theta$. A conic is determined



by the equation $r/g = e$, where e is a constant called the eccentricity. An ellipse has eccentricity $e < 1$, a parabola has eccentricity $e = 1$, and a hyperbola has eccentricity $e > 1$. A circle is not eccentric at all ($e = 0$).

Next, we derive Kepler III. What is the equation of an ellipse in Cartesian coordinates? Start with $r/g = e$:

$$\begin{aligned}
\sqrt{x^2+y^2} &= e(d-x)\\
x^2+y^2 &= e^2(x^2-2xd+d^2)\\
x^2(1-e^2)+2de^2x+y^2 &= e^2d^2\\
(1-e^2)[x^2+\frac{2de^2}{1-e^2}x]+y^2 &= e^2d^2\\
(1-e^2)[x+\frac{de^2}{1-e^2}]^2+y^2 &= e^2d^2+\frac{e^4d^2}{1-e^2}=\frac{e^2d^2}{1-e^2}\\
\frac{[x+de^2/(1-e^2)]^2}{[ed/(1-e^2)]^2}+\frac{y^2}{[ed/\sqrt{1-e^2}]^2} &= 1\\
\frac{[x+de^2/(1-e^2)]^2}{a^2}+\frac{y^2}{b^2} &= 1,
\end{aligned}$$

where $a = ed/(1-e^2)$ and $b = ed/\sqrt{1-e^2}$. This is the familiar equation for an ellipse in Cartesian coordinates. Notice that

$$b = \sqrt{a}\sqrt{ed} = \sqrt{a}\frac{2h}{c}.$$

We can compute the area of this ellipse two ways: it's πab, but it's also the integral of A' over a period T. If $A' = h$, then we get

$$\pi ab = \int_0^T A'dt = \int_0^T hdt = hT.$$

We rearrange this to read

$$T = \frac{\pi ab}{h} = \frac{\pi a\sqrt{a}}{h}\frac{2h}{\sqrt{c}} = \frac{\pi}{\sqrt{c}}a^{3/2}.$$

This is Kepler's third law.

Alright, we've shown that centripetal force with $f(r) = \frac{c}{r^2}$ implies all of Kepler's laws hold. Are there any other centripetal force laws which are also consistent with Kepler's laws? The answer is no. You can reverse all the steps we took, starting with

$$r = \frac{ed}{1 + e\cos\theta}, \qquad \frac{dA}{dt} = h$$

(respectively, the planets trace out ellipses and sweep out equal areas in equal times). Define

$$\lambda = \frac{2h}{d} \qquad c = \frac{2h\lambda}{e}$$

and start taking derivatives; eventually you'll get back to

$$\mathbf{r}" \times \hat{k} = \frac{c}{2h}\hat{l}'.$$

However, the acceleration is centripetal, so

$$\mathbf{r}" \times \hat{k} = f\frac{r^2}{2h}\hat{l}' \Rightarrow \frac{c}{2h} = f\frac{r^2}{2h} \Rightarrow f = \frac{c}{r^2}.$$

□