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Abstract. Coarse geometry deals with the large-scale geometry of a space as opposed to its small-scale structure. This paper investigates the concept of coarse geometry and specifically studies the coarse geometry of spaces regularly encountered in real analysis. We construct a non-separable space that is coarse equivalent to the separable space $L^1([a,b], m)$.

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1 Introduction

Coarse geometry is a relatively new field. The modern definitions and notions of coarse geometry that we use today were first introduced by John Roe (see [9]). In recent years, coarse geometry has been in the spotlight due to its usefulness in proving theorems related to highly celebrated conjectures. Specifically, the concept of a coarse embedding from one space to another has proven useful in the progress on the Baum-Connes and Novikov conjectures. These conjectures are two of the longstanding unsolved problems in the study of topology.

In fact in [12], Yu used coarse embeddings (See Definition 2.4) to prove a coarse version of the Baum-Connes conjecture. In [2] Gromov also conjectured that finitely generated discrete groups that are coarsely embeddable into a Hilbert space, when viewed as metric spaces, satisfy the Novikov conjecture. This was proven in [5] by Yu and Kasparov. The statements of these conjectures are beyond the scope of this paper, but they demonstrate the utility of coarse geometry.

The works stated above utilizes the concept of a coarse embedding into a Banach space. Some important results concerning embeddability include the following: In [7] it was shown by Nowak that L^p is coarsely embeddable into a Hilbert Space for $0 < p \leq 2$. Also in [8], it was shown that the Hilbert space is coarsely embeddable into any l^p for $1 \leq p < \infty$. Conversely, in [4] Johnson and Randrianarivony showed that l_p does not coarsely embed into a Hilbert space for $p > 2$. In [6](see Remark 5.10) it was proven that L^q embeds coarsely into L^p for $1 \leq q < p \leq \infty$.

In the framework of coarse geometry, we are only concerned with coarse equivalence, which is a type of coarse embedding. Despite the abundant literature on the embeddability of L^p, there are relatively few examples of metric spaces that are coarsely equivalent to $L^p([a, b])$. We shall later see that coarse equivalence is the analogous coarse geometric equivalent of the homeomorphism from topology. In view of this, we also fail to find adequate examples in the literature that show us which properties are not invariant under coarse equivalence. In turn, the goal of this paper will be two-fold. First, we will be primarily concerned in constructing a non-separable metric space that is coarse equivalent to the separable space $L^1([a, b])$. Second, this will also show that the topological property of separability is not invariant under coarse equivalence. While this may come as no surprise, as separability is a topological property, our construction will give us a tangible nontrivial example of this fact.

The paper will be structured as follows: In Section 2, we provide the necessary background on the subject at hand. We begin by defining basic definitions used in the subject of coarse geometry. We also present some examples and requisite lemmas needed to understand the rest of the paper. In Section 3.1 we give a construction of our space. This space will be shown to be non-separable and non-discrete. Then, in Section 3.2, we prove the intermediate step needed for the main result. Namely, we show that the particular non-separable space is coarse equivalent to the space of Riemann integrable functions. In Section 4.1, we introduce the necessary results concerning the space $L^p([a,b])$. These results will allow us to link the intermediate step to the main result. In Section 4.2, the main result is proven. The main result is that the particular non-separable space is coarse equivalent to $L^p([a,b])$ if and only if $p = 1$. Finally, in Section 5, we summarize our results and give an outlook to future research.

2 Background

In this section, we shall introduce the basic definitions and concepts to those unfamiliar with coarse geometry. We also incorporate several examples to illustrate how the definitions and theorems are used.

2.1 Coarse Equivalence

We introduce the notion of a coarse equivalence and give some examples. First we shall define the following:

- We define $L^p(X) = \{f : X \to \mathbb{R} \mid f \text{ is measurable and } \|f\|_{L^p} < \infty\}$ where
$$\|f\|_{L^p} = \left(\int_X |f|^p \, dm\right)^{1/p},$$
m is the lebesgue measure and $X \subset \mathbb{R}$. We refer the reader to [1] for the construction of the Lebesgue measure and integral. When the underlying set is understood, we shall just refer to this space as L^p.

- Set $\mathcal{R}^p([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f^p \text{ is Riemann integrable and } \|f\|_{\mathcal{R}^p} < \infty\}$ where
$$\|f\|_{\mathcal{R}^p} = \left(\int_a^b |f|^p \, dx\right)^{1/p},$$
and where the integral is the Riemann integral.

- If the context is clear, we shall use the notation $\|\cdot\|_p$ to mean either the norm of $\mathcal{R}^p$ or L^p.

The following definition is from [9].

Definition 2.1. Let X and Y be metric spaces, and let $f : X \to Y$ be any map.

(a) The map f is (metrically) *proper* if the inverse image, under f, of each bounded subset of Y, is a bounded subset of X.

(b) The map f is (uniformly) *bornologous* if for every $R > 0$ there is $S > 0$ such that
$$d_X(x,y) < R \Rightarrow d_Y(f(x), f(y)) < S$$
(c) The map f is *coarse* if it is proper and bornologous.

Example 2.1. Let $f : \mathbb{R}^{+} \cup \{0\} \to \mathbb{R}^{+} \cup \{0\}$ be defined as $f(x) = \sqrt{x}$. Its easy to see that f is coarse. To show bornologous assume $|x - y| < R$. If $x, y \in [0, 1]$ then $|f(x) - f(y)| \leq 1$. If $x \geq 1, y < 1$ then $\frac{1}{\sqrt{x}+\sqrt{y}} \leq 1$, hence

$$|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq |x - y| < R.$$

Similarly if $x, y \geq 1$ then $|f(x) - f(y)| < R$. Thus f is bornologous. To see that f is proper note that if S is bounded on $\mathbb{R}^{+} \cup \{0\}$, then there exists an interval such that $S \subset (a, b)$. Since f is increasing we see that $f^{-1}(S) \subset (a^2, b^2)$, which is bounded. This shows f is coarse. See figure 1.

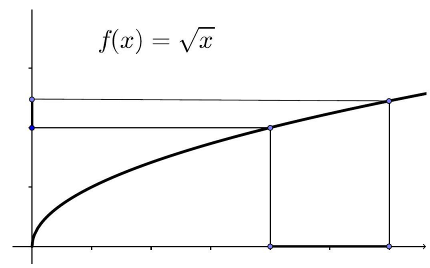


Figure 1: This shows that images and pre-images of bounded sets are bounded for $f(x) = \sqrt{x}$. Hence f is coarse.

Example 2.2. Let $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{x}$. Consider the bounded set $(0, 1)$ and note that for all $n \in \mathbb{N}$ we have $\frac{1}{n}, \frac{1}{2n} \in (0, 1)$. Now

$$\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{2n}\right) \right| = n.$$

As $n \to \infty$ we see that $\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{2n}\right) \right| \to \infty$. Thus $f((0, 1))$ is not bounded on $\mathbb{R}$, hence f is *not* bornologous and *not* coarse. See Figure 2.

Lemma 2.1. *Let (Y, d) be a metric space and $X \subset Y$. Then the inclusion map $X \hookrightarrow Y$ is coarse.*

Proof. Let $i : X \to Y$ be the inclusion. Suppose $d(x, y) < R$ for $x, y \in X$, then $d(i(x), i(y)) = d(x, y) < R$ and i is bornologous. Now, suppose $S \subset Y$ is bounded, say by R. Then if $x, y \in i^{-1}(S)$ then $d(x, y) < R$, and i is proper. Hence i is coarse. □

Definition 2.2. Two maps f, f' from a set X into a metric space Y are *close* if $d(f(x), f'(x))$ is bounded uniformly in X [9].

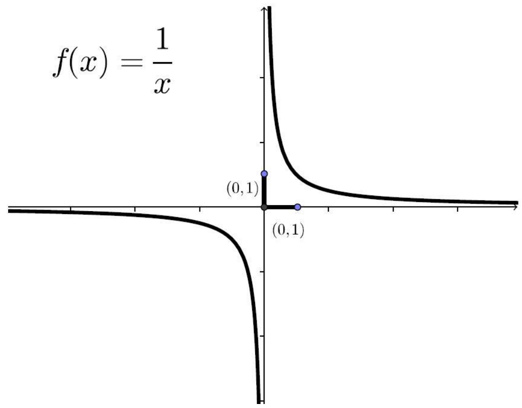


Figure 2: Note that the interval $(0, 1)$ shows that $f(x) = \frac{1}{x}$ is not coarse.

Definition 2.3. We say metric spaces X and Y are *coarse equivalent* if there exists coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X, respectively [9].

The usual intuitive notion of a coarse equivalence involves zooming out far enough on a space X, to a point where X starts to look similar to the metric space Y. In this regard, the small-scale structure (e.g. connectedness, compactness) is irrelevant when looking at the space from a large scale distance. In coarse geometry, properties such as boundedness are more relevant over topological properties which involve the characterization of open sets. In fact, one can axiomatize the theory of coarse geometry by defining a *coarse structure* as opposed to a topology on a space X [9]. In topology, if (X, d) is a metric space, then the metric

$$d'(x, y) = \min \{d(x, y), 1\}$$

induces the same topology as d. This implies that the topology on X is dependent only on the small scale structure of X. In coarse geometry, the metric

$$d''(x, y) = \max \{d(x, y), 1\}$$

induces the same coarse structure as d [11].

The following is a simple example of a coarse equivalence involving $\mathbb{R}$ and $\mathbb{Z}$.

Example 2.3. We show $(\mathbb{R}, d)$ is coarse equivalent to $(\mathbb{Z}, d)$ where $d(x, y) = |x - y|$. Define $F : \mathbb{R} \to \mathbb{Z}$ by $F(x) = \lfloor x \rfloor$ and define $G : \mathbb{Z} \to \mathbb{R}$ by $G(x) = x$. Its easy to see that F, G are coarse. Now note that for all $x \in \mathbb{Z}$ we have

$$\begin{aligned} |F \circ G\,(x) - x| &= |\lfloor x \rfloor - x| \\ &= |x - x| \\ &= 0, \end{aligned}$$

and for all $x \in \mathbb{R}$ we have

$$\begin{aligned} |G \circ F(x) - x| &= |\lfloor x \rfloor - x| \\ &< 1, \end{aligned}$$

and so $(\mathbb{R}, d)$ and $(\mathbb{Z}, d)$ are coarse equivalent [9].

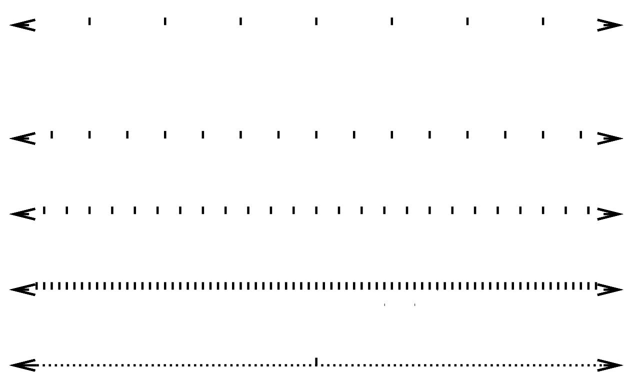


Figure 3: We can see that $\mathbb{Z}$ starts to look like $\mathbb{R}$

A heuristic is seen by observing Figure 3. Note that as we look at the integers from farther away, the points start to bunch up together. As we zoom out, it will get to a point where the integers will almost become unrecognizable from the real number line.

We conclude this section with an example.

Example. Let (X, d_X) be a bounded metric space. Let $Y = ([0, M], |\cdot|)$ where

$$M = \sup_{x,y \in X} d_X(x, y).$$

Then X is coarse equivalent to Y.

Proof. Let $y_0 \in Y$. Define $F : X \to Y$ by

$$F(x) = d_X(x, y_0)$$

and define $G : Y \to X$ by

$$G(x) = y_0.$$

We show F is coarse. Choose a bounded subset S of Y and consider $F^{-1}(S)$. Since X is bounded then $F^{-1}(S)$ must also be bounded. So F is proper. Let $x, y \in X$ satisfy $d_X(x, y) < R$, then $|d_X(x, y_0) - d_X(y, y_0)| \leq M$, and F is bornologous. Therefore F is coarse and G is also similarly coarse.

Now note that

$$\begin{aligned} d_Y\left(F \circ G\left(x\right), x\right) &= \left|d_X\left(y_0, y_0\right) - x\right| \\ &= x \\ &\leq M. \end{aligned}$$

Also,

$$\begin{aligned} d_X\left(G \circ F\left(x\right), x\right) &= d_X\left(y_0, x\right) \\ &\leq \sup_{x,y \in X} d_X\left(x, y\right) \\ &= M. \end{aligned}$$

Thus X and Y are coarse equivalent. □

2.2 Coarse Embedding

In this section we give the definition of a coarse embedding from [7]. We then give a relationship between coarse embeddings and coarse equivalences. This relationship will allow us to prove that coarse equivalence is in fact an equivalence relationship.

Definition 2.4. Let X, Y be metric spaces. A function $f : X \to Y$ is a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying
(1) $\rho_1\left(d_X\left(x, y\right)\right) \leq d_Y\left(f(x), f(y)\right) \leq \rho_2\left(d_X\left(x, y\right)\right)$ for all $x, y \in X$,
(2) $\lim_{t\to\infty} \rho_1\left(t\right) = +\infty$.

We have the following relationship between a coarse embedding and a coarse equivalence. This result has been known (see [3], [7], [9]). We include our own proof:

Proposition 2.1. *Let $h : X \to Y$ be a coarse embedding, then X and $h(X)$ are coarse equivalent.*

Proof. Let $f : X \to h(X)$ be defined as $f(x) = h(x)$ for all $x \in X$. Let ρ_1, ρ_2 be as in definition 1.4. Clearly f is onto.

Now we show f is coarse. To show bornologous, let $R > 0$ and suppose $d_X\left(x, y\right) < R$. Since ρ_2 is non-decreasing, then we can see that $\rho_2\left(d_X\left(x, y\right)\right) \leq \rho_2\left(R\right)$. Since f is a coarse embedding, we have that

$$\begin{aligned} d_Y\left(f(x), f(y)\right) &\leq \rho_2\left(d_X\left(x, y\right)\right) \\ &\leq \rho_2\left(R\right), \end{aligned}$$

hence f is bornologous. To show f is proper let $S \subset h\left(X\right)$ be bounded. Then for all $y, y' \in S$ there exists $R \geq 0$ such that $d_Y\left(y, y'\right) \leq R$. Consider the set $f^{-1}\left(S\right)$, and let $x, x' \in f^{-1}\left(S\right)$. There exists $y, y' \in S$ such that $y = f(x), y' = f(x')$. Since f is a coarse embedding we have

$$\begin{aligned} \rho_1\left(d_X\left(x, x'\right)\right) &\leq d_Y\left(f(x), f(x')\right) \\ &= d_Y\left(y, y'\right) \\ &\leq R. \end{aligned}$$

Let $R' = \sup\left\{\rho_1^{-1}\left(\{R\}\right)\right\}$, where $\rho_1^{-1}\left(\{R\}\right)$ is the pre-image of the singleton set $\{R\}$ under ρ_1. Since $\lim_{t\to\infty}\rho_1(t) = +\infty$ and ρ_1 is non-decreasing, then R' is finite. Thus $d_X(x, x') \leq R'$, proving that f is proper. This shows that f is coarse.

Now we define the function $g : f(X) \to X$. For every $y \in f(X)$ we choose $g(y)$ to be an element $x \in X$ such that $f(x) = y$. We can now show g is coarse. We begin by showing g is bornologous. Let $R > 0$ and assume $d_Y(y, y') < R$. There exists $x, x' \in X$ such that $f(x) = y, f(x') = y'$ where $g(y) = x, g(y') = x'$ since $y, y' \in f(X)$. Since f is a coarse embedding,

$$\begin{aligned}\rho_1\left(d_X(x, x')\right) &\leq d_Y\left(f(x), f(x')\right)\\ &= d_Y(y, y')\\ &< R.\end{aligned}$$

By a similar argument as before, letting $R' = \sup\left\{\rho_1^{-1}\left(\{R\}\right)\right\}$ we see that

$$\begin{aligned}d_X\left(g(y), g(y')\right) &= d_X(x, x')\\ &\leq R',\end{aligned}$$

proving that g is bornologous.

To show g is proper let $S \subset X$ be bounded. For all $x, x' \in S$ there exists $R > 0$ such that $d_Y(x, x') \leq R$. Consider the set $f^{-1}(S)$, and let $y, y' \in f^{-1}(S)$. Then there exists $x, x' \in S$ such that $x = g(y), x' = g(y')$ and by the definition of g we have that $f(x) = y$,$f(x') = y$. Since f is a coarse embedding,

$$\begin{aligned}d_Y(y, y') &= d_Y\left(f(x), f(x')\right)\\ &\leq \rho_2\left(d_X(x, x')\right)\\ &\leq \rho_2(R),\end{aligned}$$

which shows $f^{-1}(S)$ is bounded. Thus, g is proper.

To show $f \circ g$ is close to the identity in Y, note that for all $y \in f(X)$ we have

$$\begin{aligned}d_Y\left(f \circ g(y), y\right) &= d_Y\left(f(g(y)), y\right)\\ &= d_Y\left(f(x), y\right)\\ &= d_Y(y, y)\\ &= 0.\end{aligned}$$

We now show $g \circ f$ is close to the identity in X. By the definition of g, there exists $x' \in X$ such that $g(f(x)) = x'$ where $f(x) = f(x')$. We can rewrite $d_X\left(g(f(x)), x\right) = d_X(x', x)$, and since f is a coarse embedding, then $\rho_1\left(d_X(x', x)\right) \leq d_Y\left(f(x), f(x')\right) = 0$ since $f(x) = f(x')$. Again letting $C' = \sup\left\{\rho_1^{-1}\left(\{0\}\right)\right\}$ we see that $d_X(x', x) \leq C'$. We thus have shown that for all $x \in X$,

$$\begin{aligned}d_X\left(g \circ f(x), x\right) &= d_X\left(g(f(x)), x\right)\\ &= d_X(x', x)\\ &\leq C'.\end{aligned}$$

This completes the proof. □

We can in fact extend the previous result into a stronger statement. The proof relies on very similar arguments to those of Proposition 2.1.

Proposition 2.2 (See page 2,3 of [3])**.** *The spaces X and Y are coarse equivalent if and only if there exists a coarse embedding $f : X \to Y$ such that for each $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) \leq C$ for some $C > 0$ independent of y and x.*

Lemma 2.2. *Coarse equivalence is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial. We shall prove transitivity. Assume (X, d_X) is coarse equivalent to (Y, d_Y) and that (Y, d_Y) is coarse equivalent to (Z, d_Z). We shall prove that (X, d_X) is coarse equivalent to (Z, d_Z). Since X is coarse equivalent to Y, by Proposition 2.2 we can find a coarse embedding $f : X \to Y$ and non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ satisfying $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$ for all $x, y \in X$ and $\lim_{t\to\infty} \rho_1(t) = \infty$. Similarly we can find a coarse embedding $g : Y \to Z$ and non-decreasing functions $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ satisfying $\phi_1(d_Y(x, y)) \leq d_Z(g(x), g(y)) \leq \phi_2(d_Y(x, y))$ for all $x, y \in Y$ and $\lim_{t\to\infty} \phi_1(t) = \infty$. We claim that the function $h = g \circ f$ is a coarse embedding from X into Z. First note that for $x, y \in X$ we have that

$$\phi_1(d_Y(f(x), f(y))) \leq d_Z(g(f(x)), g(f(y))) \tag{1}$$

since $f(x), f(y) \in Y$. Since we know that $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y))$ by the definition of ρ_1 and since ϕ_1 is non-decreasing then we have

$$\phi_1(\rho_1(d_X(x, y))) \leq \phi_1(d_Y(f(x), f(y))). \tag{2}$$

Combining the inequality (1) with (2) we arrive at

$$\phi_1(\rho_1(d_X(x, y))) \leq d_Z(g(f(x)), g(f(y))).$$

By an analogous argument we can also show that

$$d_Z(g(f(x)), g(f(x))) \leq \phi_2(\rho_2(d_X(x, y))),$$

and so

$$\phi_1(\rho_1(d_X(x, y))) \leq d_Z(g(f(x)), g(f(y))) \leq \phi_2(\rho_2(d_X(x, y))).$$

Notice that we also have that $\lim_{t\to\infty} \phi_1(\rho_1(t)) = \infty$, $\phi_1 \circ \rho_1, \phi_2 \circ \rho_2$ are non-decreasing. Thus the function $g \circ f$ satisfies the conditions of a coarse embedding from X into Z.

If we can show that for all $z \in Z$ there exists $x \in X$ such that $d_Z(g \circ f(x), z) \leq C$ for some $C \geq 0$ independent of z and x, then by Proposition 2.2 X is coarse equivalent to Z. By Proposition 2.2, there exists constants $C_1, C_2 \geq 0$ such that for all $y \in Y, z \in Z$ there exists $x_y \in X, y_z \in Y$ such that $d_Y(f(x_y), y) \leq C_1$ and $d_Z(g(y_z), z) \leq C_2$. Choose $x \in X$ such that $d_Y(f(x), y_z) \leq C_1$ and note that

$$\begin{aligned} d_Z(g \circ f(x), z) &\leq d_Z(g(f(x)), g(y_z)) + d_Z(g(y_z), z) \\ &\leq \phi_2(d_Y(f(x), y_z)) + C_2 \\ &\leq \phi_2(C_1) + C_2. \end{aligned}$$

This completes the proof since this is valid for any $z \in Z$. □

3 A metric space that is coarse equivalent to $\mathcal{R}^1([a,b])$

3.1 The metric space $\mathcal{P}([a,b])$

In this section, we construct a metric space that is coarse equivalent to $\mathcal{R}^1([a,b])$. This metric space arises quite naturally. We shall first define this metric space. We then give the necessary preliminary lemmas to prove the main theorem. First we recall that a *partition* P of $[a,b]$ is an ordered set $P = \{a = x_0, x_1, \ldots x_{n-1}, x_n = b\}$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$.

Definition 3.1. A *partition function* f_P of an interval $[a,b]$ is a function $f_P : P \to \mathbb{R}$ where P is a partition of $[a,b]$.

Notation 3.1. We set $\mathcal{P}([a,b])$ to be the set of all partition functions $f_P : P \to \mathbb{R}$ on $[a,b]$ where P ranges over all partitions of $[a,b]$.

Definition 3.2. Let f_P be a partition function on $[a,b]$. Define $Lin : \mathcal{P}([a,b]) \to \mathcal{R}^1([a,b])$ by

$$Lin(f_P) := \begin{cases} f_P(x_i) & , x = x_i \\ \frac{f_P(x_i) - f_P(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}) + f_P(x_{i-1}) & , x \in (x_{i-1}, x_i) \end{cases}.$$

Lemma 3.1. *Define* $d_{\mathcal{P}} : \mathcal{P}([a,b]) \times \mathcal{P}([a,b]) \to \mathbb{R}$ *for* $f_{P_1}, g_{P_2} \in \mathcal{P}([a,b])$ as

$$d_{\mathcal{P}}(f_{P_1}, g_{P_2}) = \begin{cases} \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx & , P_1 = P_2 \\ \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx + 1 & , P_1 \neq P_2 \end{cases},$$

then $(\mathcal{P}([a,b]), d_{\mathcal{P}})$ *is a metric space.*

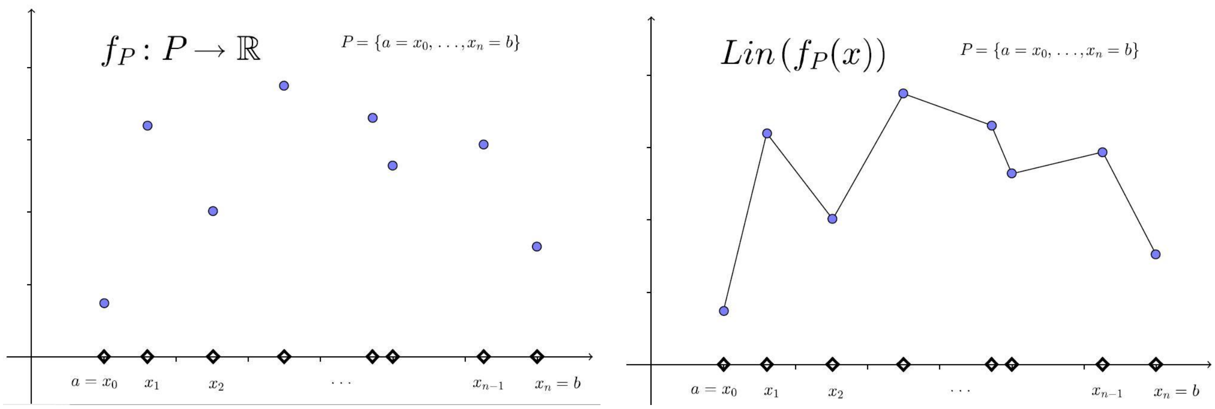
Proof. It is obvious that $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) \geq 0$ for all $f_{P_1}, g_{P_2} \in \mathcal{P}([a,b])$. Observe that if $P_1 \neq P_2$ then $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) \neq 0$. Suppose that $f_{P_1} = g_{P_2}$. Since this means that $P_1 = P_2$ and $Lin(f_{P_1}) = Lin(g_{P_2})$ then we have that $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) = 0$. This shows that $d_{\mathcal{P}}$ is positive definite. From the definition of $d_{\mathcal{P}}$, symmetry is obvious.

We now prove the triangle inequality: $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) \leq d_{\mathcal{P}}(f_{P_1}, h_{P_3}) + d_{\mathcal{P}}(h_{P_3}, g_{P_2})$ if $f_{P_1}, g_{P_2}, h_{P_3} \in \mathcal{P}([a,b])$. The triangle inequality follows from the properties of absolute values and the fact that $\int_a^b |f|dx \geq 0$ for any function $f \in \mathcal{R}^1([a,b])$. We consider the case when $P_1 \neq P_2 = P_3$. Notice that

$$\begin{aligned} d_{\mathcal{P}}(f_{P_1}, g_{P_2}) &= \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx + 1 \\ &\leq \int_a^b |Lin(f_{P_1})(x) - Lin(h_{P_3})(x)|\, dx + 1 + \int_a^b |Lin(h_{P_3})(x) - Lin(g_{P_2})(x)|\, dx \\ &= d_{\mathcal{P}}(f_{P_1}, h_{P_3}) + d_{\mathcal{P}}(h_{P_3}, g_{P_2}). \end{aligned}$$

The proof in all other possibilities for the inequalities of P_1, P_2 and P_3 are done similarly. This proves the triangle inequality and proves $(\mathcal{P}([a,b]), d_{\mathcal{P}})$ as a metric space. □

For a visualization of the elements in $\mathcal{P}([a,b])$ refer to Figure 4(a).



(a) A Partition Function (b) The *Lin* Function

Figure 4: The graph of a partition function and its *Lin* image.

Lemma 3.2. *The function $Lin(f_P)$ is coarse.*

Proof.

Lin(f_p) is bornologous:

To prove *Lin* is bornologous, consider all $f_P, g_P \in \mathcal{P}([a,b])$ such that $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) < M$. Note that if $P_1 = P_2 = P$ then $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) = \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx < M$ implies

$$\begin{aligned} \|Lin(f_P) - Lin(g_P)\|_1 &= \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx \\ &< M. \end{aligned}$$

If $P_1 \neq P_2$ then for all $f_{P_1}, g_{P_2} \in \mathcal{P}([a,b])$ such that $d_{\mathcal{P}}(f_{P_1}, g_{P_2}) = \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx + 1 < M$ then

$$\begin{aligned} \|Lin(f_P) - Lin(g_P)\|_1 &= \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx \\ &< \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx + 1 \\ &< M, \end{aligned}$$

which proves that *Lin* is bornologous.

Lin(f_p) is proper:

Now take $S \subset \mathcal{R}^1([a,b])$ where S is bounded. Thus there exists M such that $\|f - g\|_1 = \int_a^b |f(x) - g(x)|\, dx \leq M$ for all $f, g \in S$. For every $f_P \in Lin^{-1}(S)$ with partition P, there exists $f \in S$ such that $f = Lin(f_P)$. Thus for all $f_{P_1}, g_{P_2} \in Lin^{-1}(S)$ such that $P_1 = P_2 = P$, we have

$$\begin{aligned} d_{\mathcal{P}}(f_P, g_P) &= \int_a^b |Lin(f_{P_1})(x) - Lin(g_{P_2})(x)|\, dx \\ &= \int_a^b |f(x) - g(x)|\, dx, \text{ where} f, g \in S \\ &\leq M. \end{aligned}$$

Suppose $f_{P_1}, g_{P_2} \in Lin^{-1}(S)$ and that $P_1 \neq P_2$, then using the same argument we get that

$$\begin{aligned} d_{\mathcal{P}}(f_P, g_P) &= \int_a^b |Lin(f_{P_1}) - Lin(g_{P_2})|\, dx + 1 \\ &= \int_a^b |f(x) - g(x)|\, dx + 1, \text{ where} g, f \in S \\ &\leq M + 1. \end{aligned}$$

We have shown that *Lin* is proper. Since *Lin* is both proper and bornologous then *Lin* is coarse. □

We recall from topology that a metric space X is said to be *separable* if it contains a countable dense subset.

Theorem 3.1. *$\mathcal{P}([a,b])$ is not separable.*

Proof. To show $\mathcal{P}([a,b])$ is not separable it suffices to construct an uncountable set without any limit points. Such a set would negate the existence of a countable dense set.

Define $\mathcal{K}$ to be the set of all partitions from a to b. Clearly K is uncountable. For each distinct partition $P \in \mathcal{K}$, define the partition function $g_P : P \to \mathbb{R}$ by $g_P(x) = 1$ for all $x \in P$. Let $A = \{g_P\}_{P \in \mathcal{K}}$ and note that A is uncountable since there are uncountable many partitions. By the definition of g_P we have that $d_{\mathcal{P}}(g_{P'}, g_{P''}) \geq 1$ if and only if $g_{P'} \neq g_{P''}$ since every distinct element of A is defined on a distinct partition. If D is any dense set in $\mathcal{P}([a,b])$ then for each $g_P \in A$ the ball of radius $\frac{1}{2}$ centered at g_P intersects $D \setminus \{g_P\}$. But since each $g_P \in A$ is at least a distance of 1 apart, this implies that D is uncountable. Therefore $\mathcal{P}([a,b])$ is not separable. □

Now that we have defined the non-separable space $\mathcal{P}([a,b])$, it's instructive to discuss the topology induced by its metric. The metric $d_{\mathcal{P}}$ is quite similar to the metric that induces the discrete topology, namely

$$d(x,y) = \begin{cases} 1 & , x \neq y \\ 0 & , x = y \end{cases}.$$

Therefore one might ask whether the topology on $\mathcal{P}([a,b])$ is the discrete topology, but we shall see that the answer is no. To show this we recall from topology that if a space contains a limit point then its topology is not discrete. For if x is a limit point of a space X, then every neighborhood of x contains another point y of X. The set $\{x\}$ is not open in X since it contains only one point. Therefore X is not discrete since all singletons are open in the discrete topology.

To show $(\mathcal{P}([a,b]), d_{\mathcal{P}})$ does not have the discrete topology, we find a limit point. First we fix a partition P of $[a,b]$ and define the partition function $f_P : P \to \mathbb{R}$ by $f_P(x) = 0$ for all $x \in P$. Next, define the partition functions $f_{P_n} : P \to \mathbb{R}$ by $f_{P_n}(x) = \frac{1}{n}$ for all $x \in P$. Now notice that

$$\begin{aligned} d_{\mathcal{P}}(f_{P_n}, f_P) &= \int_a^b |Lin(f_{P_n})(x) - Lin(f_P)(x)|\,dx \\ &= \int_a^b \left|\frac{1}{n}\right| dx \\ &= \frac{b-a}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This shows that for a fixed partition P, f_P is a limit point and by the above discussion the metric $d_{\mathcal{P}}$ does not induce the discrete topology. The fact that $\mathcal{P}([a,b])$ is not discrete makes this space more interesting. As we will see in the proofs of the main theorems, the arguments come from a basic understanding of integration theory. This shows that $\mathcal{P}([a,b])$ arises naturally in the study of analysis. Moreover, the partition functions can be seen as functions that approximate $L^1([a,b])$ functions. In turn, they can be viewed as an alternative to simple functions.

3.2 The Intermediate Step

Before we prove the main theorem of the paper, we need one preliminary lemma and an intermediate result that will be used in the proof of the main theorem. Suppose f is a bounded real function on $[a,b]$. Corresponding to each partition P, let

$$\begin{aligned}
\Delta x_i &= x_i - x_{i-1} \quad \{i = 1, ..., n\} \\
M_i &= \sup_{x \in [x_{i-1}, x_i]} f(x) \qquad \{i = 1, ..., n\}, \\
m_i &= \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad \{i = 1, ..., n\}, \\
U(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i, \\
L(f, P) &= \sum_{i=0}^{n} m_i \Delta x_i.
\end{aligned}$$

If P is a partition of $[a, b]$, then the mesh $\|P\|$ of P is defined to be

$$\|P\| = \max_{1 \leq i \leq n} |x_i - x_{i-1}|.$$

The following gives a condition for Riemann integrability.

Lemma 3.3 ([10], Theorem 6.6)**.** *$f \in \mathcal{R}^1([a, b])$ if and only if for every $\epsilon > 0$ there exists a partition P such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Lemma 3.4. *Let $\{P_n\}_1^\infty$ be a family of partitions of $[a, b]$ such that*

$$\lim_{n \to \infty} \|P_n\| = 0.$$

If $f \in \mathcal{R}^1([a, b])$, then for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$U(f, P_n) - L(f, P_n) < \epsilon.$$

Proof. Since $f \in \mathcal{R}^1([a, b])$, f is bounded and there exists M such that $|f(x)| \leq M$. Let $\epsilon > 0$. By Lemma 3.3, there exists a partition W such that $U(f, W) - L(f, W) < \frac{\epsilon}{2}$. Let B be the number of elements in W. Since $\|P_n\| \to 0$, let N be such that

$$\|P_n\| < \min\left\{\frac{\epsilon}{8BM}, \min_{\substack{1 \leq i \leq n \\ x_i \in W}} |x_i - x_{i-1}|\right\}$$

for $n \geq N$. Let $P' = P_n \cup W$ so that we adjoin at most B points to the partition P_n. Since P' is a refinement of P_n, then $L(f, P') \geq L(f, P_n)$ and $U(f, P_n) \geq U(f, P')$ (see page 123 of [10]). Hence $L(f, P') - L(f, P_n) \geq 0$ and $U(f, P_n) - U(f, P') \geq 0$. We wish to show that

$$L(f, P') - L(f, P_n) \leq 2MB \|P_n\|. \tag{3}$$

Let $[x_{i-1}, x_i]$ be an interval of P_n where one of the points of W falls inside this interval, say $x' \in W$. Let

$$\begin{aligned} m_i &= \inf f(x) \qquad (x_{i-1} \leq x \leq x_i), \\ m' &= \inf f(x) \qquad (x_{i-1} \leq x \leq x'), \\ m'' &= \inf f(x) \qquad (x' \leq x \leq x_i), \end{aligned}$$

and let $m''' = \max\{m', m''\}$. Notice that we can show that

$$\begin{aligned} L(f, \{x_{i-1}, x', x_i\}) - L(f, \{x_{i-1}, x_i\}) &= m'(x' - x_{i-1}) + m''(x_i - x') - m_i(x_i - x_{i-1}) \\ &\leq m'''(x' - x_{i-1}) + m'''(x_i - x') - m_i(x_i - x_{i-1}) \\ &= m'''(x_i - x_{i-1}) - m_i(x_i - x_{i-1}) \\ &= (m''' - m_i)(x_i - x_{i-1}) \\ &\leq 2M\|P_n\|. \end{aligned}$$

Since we are adjoining at most B points to P_n, we can repeat this same reasoning to each interval of P_n containing the B points of W and arrive at (3). Similarly we can show that

$$U(f, P_n) - U(f, P') \leq 2MB\|P_n\|.$$

We can now conclude that

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= [U(f, P_n) - U(f, P')] + [L(f, P') - L(f, P_n)] \\ &\quad + [U(f, P') - L(f, P')] \\ &\leq 4BM\|P_n\| + [U(f, P') - L(f, P')] \\ &\leq 4BM\|P_n\| + [U(f, W) - L(f, W)] \\ &< 4BM\frac{\epsilon}{8BM} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

We are now ready to prove our result concerning a metric space that is coarse equivalent to $\mathcal{R}^1([a,b])$. This theorem is the intermediate step we need for us to show that $\mathcal{P}([a,b])$ is coarse equivalent to the separable space $L^1([a,b])$. Once we prove the following result, the main result will follow readily.

Theorem 3.2. *$(\mathcal{P}([a,b]), d_{\mathcal{P}})$ is coarse equivalent to $(\mathcal{R}^1([a,b]), \|\cdot\|_1)$.*

Proof. Let $\epsilon > 0$. We shall use $Lin : \mathcal{P}([a,b]) \to \mathcal{R}^1([a,b])$ in one direction of the proof. Let $P_n = \left\{a + \frac{b-a}{n} i\right\}_{i=0}^{n}$ and define $G : \mathcal{R}^1([a,b]) \to \mathcal{P}([a,b])$ as

$$G(f) = f_{P_N},$$

where f_{P_N} is a partition function on P_N defined as $f_{P_N}(x) = f(x)$ for all $x \in P_N$ and where

$$N = \min\left\{n \in \mathbb{N} \mid U(f, P_k) - L(f, P_k) < \epsilon, \forall k \geq n\right\}.$$

By Lemma 3.4 and the well ordering of $\mathbb{N}$ we have the existence of such an N. Recall that Lin is coarse by Lemma 3.2. We show G is coarse.

G is proper:

Consider a bounded set $S \subset \mathcal{P}([a,b])$, thus for all $f_P, g_{P'} \in S$ there exists an M such that $d_{\mathcal{P}}(f_P, g_{P'}) \leq M$. For all $f \in G^{-1}(S) \subset \mathcal{R}^1([a,b])$ there exists a $f_{P_N} \in S$ such that $f(x_i) = f_{P_N}(x_i)$ for $x_i \in P_N$ and where $x_i = a + \frac{b-a}{N} i$. By the definition of G this means that

$$\inf_{x \in [x_{i-1}, x_i]} f(x) \leq Lin\left(f_{P_N}\right)(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$\inf_{x \in [x_{i-1}, x_i]} f(x) \leq f(x) \leq \sup_{x \in [x_{i-1}, x_i]} f(x)$$

for all $x \in [x_{i-1}, x_i]$. By putting these together we arrive at

$$\left|Lin\left(f_{P_N}\right)(x) - f(x)\right| \leq \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x), \tag{4}$$

when $x \in [x_{i-1}, x_i]$.

Now let $f, g \in G^{-1}(S)$. Suppose the corresponding image functions $f_{P_{N_1}}, g_{P_{N_2}}$, have different partitions, say $P_{N_1} = \{x_i\}_{i=1}^{N_1}$ and $P_{N_2} = \{y_i\}_{i=1}^{N_2}$. Since

$$d_{\mathcal{P}}(f_{P_{N_1}}, g_{P_{N_2}}) = \int_a^b \left|Lin\left(f_{P_{N_1}}\right)(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx + 1 < M, \tag{5}$$

then,

$$\begin{aligned}
\|f-g\|_1 &\leq \left\|f - Lin\left(f_{P_{N_1}}\right)\right\|_1 + \left\|Lin\left(f_{P_{N_1}}\right) - Lin\left(g_{P_{N_2}}\right)\right\|_1 \\
&\quad + \left\|Lin\left(g_{P_{N_1}}\right) - g\right\|_1 + 1 \\
&= \int_a^b \left|f(x) - Lin\left(f_{P_{N_1}}\right)(x)\right| dx \\
&\quad + \int_a^b \left|Lin\left(f_{P_{N_1}}\right)(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx + \int_a^b \left|Lin\left(g_{P_{N_1}}\right)(x) - g(x)\right| dx + 1 \\
&= \sum_{i=1}^{N_1} \int_{x_{i-1}}^{x_i} \left|f(x) - Lin\left(f_{P_{N_1}}\right)(x)\right| dx + \sum_{i=1}^{N_2} \int_{y_{i-1}}^{y_i} \left|g(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx \\
&\quad + \int_a^b \left|Lin\left(f_{P_{N_1}}\right)(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx + 1 \\
&\leq \sum_{i=1}^{N_1} \int_{x_{i-1}}^{x_i} \left|\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)\right| dx \\
&\quad + \sum_{i=1}^{N_2} \int_{y_{i-1}}^{y_i} \left|\sup_{x\in[y_{i-1},y_i]} g(x) - \inf_{x\in[y_{i-1},y_i]} g(x)\right| dx + M \qquad \text{by Inequality (4) and (5),} \\
&= \sum_{i=1}^{N_1} \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)\right) \Delta x_i \\
&\quad + \sum_{i=1}^{N_2} \left(\sup_{x\in[y_{i-1},y_i]} g(x) - \inf_{x\in[y_{i-1},y_i]} g(x)\right) \Delta y_i + M \\
&= U(f,P_{N_1}) - L(f,P_{N_1}) + U(g,P_{N_2}) - L(g,P_{N_2}) + M \\
&< 2\epsilon + M.
\end{aligned}$$

If $P_{N_1} = P_{N_2}$ then the bound of $\|f-g\|_1 < 2\epsilon + M$ is found similarly. So G is proper.

G is bornologous:

To show G is bornologous suppose $f, g \in \mathcal{R}^1([a,b])$ and that $G(f)$ is defined on P_{N_1} and $G(g)$ is defined on P_{N_2}. Assume that $P_{N_1} \neq P_{N_2}$, so if $\|f-g\|_1 = \int_a^b |f(x)-g(x)|\,dx < M$, then

$$\begin{aligned}
d_{\mathcal{P}}\left(G(f), G(g)\right) &= \int_a^b \left|Lin\left(f_{P_{N_1}}\right)(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx + 1 \\
&\leq \int_a^b \left|Lin\left(f_{P_{N_1}}\right)(x) - f(x)\right| dx + \int_a^b |f(x)-g(x)|\,dx \\
&\quad + \int_a^b \left|g(x) - Lin\left(g_{P_{N_2}}\right)(x)\right| dx + 1,
\end{aligned}$$

and by similar arguments in the discussion of G being proper we have that

$$d_{\mathcal{P}}\left(G(f), G(g)\right) < 2\epsilon + M + 1.$$

If $P_{N_1} = P_{N_2}$ then the bound of $d_Y(f, g) < 2\epsilon + M$ is found similarly. So G is bornologous and proper and thus G is coarse.

$Lin \circ G$ and $id_{\mathcal{R}^1}$ are close:

We prove $Lin \circ G\,(f)$ and f are close in $\mathcal{R}^1\left([a, b]\right)$. Note that $Lin \circ G\,(f)$ is defined such that

$$\inf_{x\in[x_{i-1},x_i]} f(x) \leq Lin \circ G\,(f)\,(x) \leq \sup_{x\in[x_{i-1},x_i]} f(x)$$

for $x \in [x_{i-1}, x_i]$ and where P_N is the partition of $G\,(f)$. We also have

$$\inf_{x\in[x_{i-1},x_i]} f(x) \leq f(x) \leq \sup_{x\in[x_{i-1},x_i]} f(x)$$

on the same intervals. Putting this together yields

$$\left|Lin \circ G\,(f)\,(x) - f(x)\right| \leq \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)$$

for $x \in [x_{i-1}, x_i]$. Thus

$$\begin{aligned}
\left\|Lin \circ G\,(f) - f\right\|_1 &= \int_a^b \left|Lin \circ G\,(f)\,(x) - f(x)\right| dx \\
&= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left|Lin \circ G\,(f)\,(x) - f(x)\right| dx \\
&\leq \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)\right) dx \\
&= \sum_{i=1}^{N} \sup_{x\in[x_{i-1},x_i]} f(x)\Delta x - \sum_{i=1}^{N} \inf_{x\in[x_{i-1},x_i]} f(x)\Delta x \\
&= U(f, P) - L(f, P) \\
&< \epsilon,
\end{aligned}$$

which shows that $Lin \circ G$ and $id_{\mathcal{R}^1}$ are close in $\mathcal{R}\left([a, b]\right)$.

$G \circ Lin$ and $id_{\mathcal{P}}$ are close:

We now prove that $G \circ Lin$ and $\mathrm{id}_{\mathcal{P}}$ are close in $\mathcal{P}\left([a, b]\right)$. Suppose $G \circ Lin\,(f_P)$ is defined on a partition P_N. Also note that

$$\min_{x\in[x_{i-1},x_i]} Lin\,(f_P)\,(x) \leq Lin\,(G \circ Lin\,(f_P))\,(x) \leq \max_{x\in[x_{i-1},x_i]} Lin\,(f_P)\,(x)$$

for $x \in [x_{i-1}, x_i]$. We also have

$$\min_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x) \leq Lin\left(f_P\right)(x) \leq \max_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x)$$

on the same intervals and putting it together gives us

$$\left|Lin\left(G \circ Lin\left(f_P\right)\right)(x) - Lin\left(f_P\right)(x)\right| \leq \max_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x) - \min_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x) \quad (6)$$

for $x \in [x_{i-1}, x_i]$. By the definition of G we also have that

$$U\left(Lin\left(f_P\right), P_N\right) - L\left(Lin\left(f_P\right), P_N\right) < \epsilon. \quad (7)$$

We conclude that

$$\begin{aligned}
d_{\mathcal{P}}(G \circ Lin\left(f_P\right), f_P) &\leq \int_a^b \left|Lin\left(G \circ Lin\left(f_P\right)\right)(x) - Lin\left(f_P\right)(x)\right| dx + 1 \\
&\leq \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(\max_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x) - \min_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x)\right) dx + 1, \text{by } (6) \\
&= \sum_{i=1}^{N} \left(\max_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x) - \min_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x)\right) \Delta x_i + 1 \\
&= \sum_{i=1}^{N} \left(\max_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x)\right) \Delta x_i - \sum_{i=1}^{N} \left(\min_{x\in[x_{i-1},x_i]} Lin\left(f_P\right)(x)\right) \Delta x_i + 1 \\
&= U\left(Lin\left(f_P\right), P_N\right) - L\left(Lin\left(f_P\right), P_N\right) + 1, \text{ by } (7) \\
&< \epsilon + 1.
\end{aligned}$$

Thus $G \circ Lin$ and $\mathrm{id}_{\mathcal{P}}$ are close in $\mathcal{P}\left([a, b]\right)$ which completes the proof. □

4 A metric space that is coarse equivalent to $L^p\left([a, b]\right)$

4.1 The space $L^p\left([a, b]\right)$

In this section we prove some results involving the coarse geometry of $L^p\left([a, b]\right)$. We refer the reader to Folland ([1], pg 181) for the basic theory of L^p spaces. We remind the reader of the following definitions, which are from [1]:

Definition 4.1. A *simple function* $\phi : X \to \mathbb{R}$ on X is of the form $\phi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$, where $a_j \in \mathbb{R}$ and χ_A is the indicator function on a set A. We also require that E_j be disjoint measurable sets on X such that $X = \cup_{j=1}^{n} E_j$.

Definition 4.2. A *step function* $\psi : X \to \mathbb{R}$ on $X \subset \mathbb{R}$ is of the form $\psi(x) = \sum_{j=1}^{n} a_j \chi_{I_j}(x)$, where $a_j \in \mathbb{R}$ and χ_{I_j} is the indicator function on an interval I_j. We also require that the intervals I_j be disjoint on X such that $X = \cup_{j=1}^{n} E_j$.

We first need the following results about $L^p([a,b])$. They are standard results in Real Analysis .

Lemma 4.1 ([1], Proposition 6.7)**.** *For $1 \leq p < \infty$, the set of simple functions $\phi = \sum_1^n a_j \chi_{E_j}$, where $m(E_j) < \infty$ for all j, is dense in $L^p([a,b])$.*

Proposition 4.1 ([1], see page 187)**.** *$L^p([a,b])$ is separable for $1 \leq p < \infty$. Moreover, there exists a countable dense subset of simple functions in $L^p([a,b])$.*

Separability in $L^p([a,b])$ is usually proven by finding a countable dense set of simple functions in view of Lemma 4.1. Equally well known but less widely articulated is the density of the step functions. In fact, one can also find a countable dense set of step functions using straight forward arguments that follow from Proposition 4.1. Thus we have the following:

Lemma 4.2. *There exists a countable dense set of step functions in $L^p([a,b])$ for $1 \leq p < \infty$.*

Theorem 4.1. *$\mathcal{R}^p([a,b])$ is coarse equivalent to $L^p([a,b])$.*

Proof. Let $\epsilon > 0$. Let $F : \mathcal{R}^p([a,b]) \to L^p([a,b])$ be the inclusion. Now let $A = \{\phi_n\}_1^\infty$ be a countable dense set of step functions by Lemma 4.2. Define $G : L^p([a,b]) \to \mathcal{R}^p([a,b])$ by

$$G(f) = \phi_N$$

where

$$N = \min\{n \in \mathbb{N} \mid \|f - \phi_n\|_{L^p} < \epsilon\}.$$

Its easy to see that G is well defined since A is dense in $L^p([a,b])$ and by the well ordering of $\mathbb{N}$. Also recall that since step functions are Riemann integrable then $A \subset \mathcal{R}^p([a,b])$. We see that F is coarse by Lemma 2.1. We now show G is coarse. To show bornologous, suppose $\|f - g\|_{L^p} < R$ for $f, g \in L^p([a,b])$. First note that $G(f) = \phi_N$ and $G(g) = \phi_M$ for some $N, M \in \mathbb{N}$. We see that

$$\begin{aligned} \|G(f) - G(g)\|_{\mathcal{R}^p} &= \|\phi_N - \phi_M\|_{\mathcal{R}^p} \\ &= \|\phi_N - \phi_M\|_{L^p}, \text{since }, \phi_N, \phi_M \in L^p \\ &\leq \|\phi_N - f\|_{L^p} + \|f - g\|_{L^p} + \|g - \phi_M\|_{L^p} \\ &< R + 2\epsilon. \end{aligned}$$

This shows G is bornologous. We can similarly show G is proper, hence G is coarse. Now note that

$$\begin{aligned} \|F \circ G(f) - f\|_{L^p} &= \|\phi_N - f\|_{L^p} \\ &< \epsilon, \end{aligned}$$

and so $F \circ G(f)$ is close to id_{L^p}. We also have

$$\begin{aligned}
\|G \circ F(f) - f\|_{\mathcal{R}^p} &= \|G(f) - f\|_{\mathcal{R}^p} \\
&= \|\phi_N - f\|_{\mathcal{R}^p}, \text{for some } \phi_N, \\
&= \|\phi_N - f\|_{L^p}, \text{since } \phi_N, f \in \mathcal{R}^p \subset L^p, \\
&< \epsilon,
\end{aligned}$$

and so $G \circ F(f)$ is close to $\mathrm{id}_{\mathcal{R}^p}$. This completes the proof. □

4.2 Main Result

Before proving the main result of the paper, it is also an interesting question to ask for which pairs of $p, q \geq 1$ is $L^p(X, \mu)$ coarse equivalent to $L^q(X, \mu)$. In fact Nowak showed that this never occurs for different $p's$ and $q's$. Its proof is beyond the scope of this paper.

Lemma 4.3 ([8], pages 115-16)**.** *Let (X, μ) be a measure space such that $L^p(X, \mu)$ and $L^q(X, \mu)$ are separable. If $p \neq q$ then $L^p(X, \mu)$ is not coarse equivalent to $L^p(X, \mu)$.*

We have shown in Theorem 3.1 that $\mathcal{P}([a, b])$ was non-separable and Proposition 4.1 tells us that $L^p([a, b])$ is separable. The next theorem shows us that the topological property of separability is not invariant under coarse equivalence.

Theorem 4.2. *$\mathcal{P}([a, b])$ is coarse equivalent to $L^p([a, b])$ if and only if $p = 1$.*

Proof. Assume $p = 1$. Using Theorem 3.2, Theorem 4.1 and Lemma 2.2 we get that $\mathcal{P}([a, b])$ is coarse equivalent to $L^p([a, b])$.

Conversely suppose that $\mathcal{P}([a, b])$ is coarse equivalent to $L^p([a, b])$. Since $\mathcal{P}([a, b])$ is coarse equivalent to $L^1([a, b])$ this implies that $L^1([a, b])$ is also coarse equivalent to $L^p([a, b])$ by Lemma 2.2. By taking the contrapositive of Lemma 4.3, since $L^p([a, b])$ is coarse equivalent to $L^1([a, b])$ then $p = 1$.

□

5 Conclusion and Future Work

We have given a basic introduction to the coarse geometry of spaces that are normally encountered in real analysis. We did this using the important tool of coarse equivalence to show that the coarse geometry of $L^p([a, b])$ is equivalent to the coarse geometry of $\mathcal{R}^p$. We have also shown that separability is not invariant in coarse geometry by constructing a non-separable metric space that is coarse equivalent to the separable space $L^p([a, b])$. Our work presents several examples of coarse equivalences of classical spaces.

It is also interesting to note that our work also adds to the list of pathological properties of $L^1([a, b])$. Our theorem showed that $\mathcal{P}([a, b])$ is not coarse equivalent to $L^p([a, b])$ for

$p > 1$. Therefore an immediate question that follows is if there exists a non-separable space which is coarse equivalent to $L^p([a,b])$ when $1 < p < \infty$. For future research, we wish to find sufficient and necessary conditions for a space X to be coarse equivalent to $L^p([a,b])$. Our work considered L^p equipped only with the Lebesgue measure on $[a,b]$. We wish to further extend our work to general measurable spaces. We also wish to further study the coarse geometry of $L^2([a,b])$ more in depth since it is a Hilbert space.

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