Gradient Estimates on Manifolds Using Coupling for Diffusion Processes

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Introduction

- Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two probability spaces. A *coupling* of μ_1 and μ_2 is a measure μ on the product space satisfying the marginality condition: For all $A \in \mathcal{F}_1, B \in \mathcal{F}_2$

$$\begin{aligned} \mu\left(A \times \Omega_2\right) &= \mu_1\left(A\right), \\ \mu\left(\Omega_1 \times B\right) &= \mu_2\left(B\right). \end{aligned}$$

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- Our approach to coupling will be to couple two diffusion processes.

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$$dX_t = \sigma(t, X_t)\, dB_t + b(t, X_t)\, dt, \qquad X_0 = x$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, and $B_t = \left(B_t^1, \ldots, B_t^d\right)$ is a Brownian motion.

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  - Its generator L is defined by

    $$L = \frac{1}{2} \sum_{i,j=1}^{d} \left(\sigma\sigma^T\right)^{(ij)}(x) \frac{\partial^2}{\partial x^{(i)} \partial x^{(j)}} + \sum_{i=1}^{d} b^{(i)}(x) \frac{\partial}{\partial x^{(i)}}.$$

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  - This is a second order differential operator.

- Let M be a manifold. In terms of diffusion processes, a **coupling** is a pair of diffusions (X_t, Y_t) with $\mathbb{P}^{(x,y)}(X_0 = x, Y_0 = y) = 1$ such that under the measure $\mathbb{P}^{(x,y)}$ the marginal processes X and Y are both diffusions on M with the same generator L.
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    $$dX_t = \sigma(t, X_t)\, dB_t + b(X_t)\, dt, \qquad X_0 = x,$$

    where $(B_t)_{t\geq 0}$ is a Brownian motion. The idea is to find a new Brownian motion B_t' such that Y_t solves

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    on the same probability space, having the same distribution of X_t.
  - So the only thing we have to do is choose a suitable Brownian motion B_t'.
- **Example:** (Classical coupling)
  - In this case B_t' is a new Brownian motion, independent of B_t.

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- A coupling is said to be **successful** if

$$\mathbb{P}(T < \infty) = 1.$$

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  - Set $u(t,x,y) = \frac{\sigma(t,y)^{-1}(x-y)}{\left|\sigma(t,y)^{-1}(x-y)\right|}$ and

$$H(t,x,y) = I - 2u(t,x,y)\, u(t,x,y)^T$$

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  - H is reflection in the plane orthogonal to $\sigma(t,y)^{-1}(x-y)$.

Coupling by Reflection

Theorem (Lindvall-Rogers, 1986)

Suppose σ, b *are Lipschitz and bounded, and that*

$$\Lambda \equiv \sup_x \left\|\sigma^{-1}(x)\right\| < \infty.$$

Suppose also that

$$\sup_{x,y} \|\sigma(x) - \sigma(y)\| < 2\Lambda^{-1}.$$

If

$$\int_1^\infty dr \left\{ \exp\left[-2 \int_1^r \gamma(u) du \right] \right\} = \infty$$

where γ *is defined in a certain way in terms of* $\alpha = \sigma(x) - \sigma(y) H(t,x,y)$ *, then we have a successful coupling* X *and* Y*. That is,* $\mathbb{P}(T < \infty) = 1$.

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- Recall the reflection coupling

$$dX_t = \sigma(t, X_t)\, dB_t + b(X_t)\, dt, \qquad X_0 = x,$$

where we construct

$$dY_t = \sigma(t, Y_t)\, dB'_t + b(Y_t)\, dt, \qquad Y_0 = y,$$

where $dB'_t = H_t dB_t$ and $H_t = H(t, X_t, Y_t)$ was chosen to be reflection in the plane orthogonal to $\sigma(t, y)^{-1}(x - y)$.

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    Thus the coupling time T is the first time B_t hits the hyperplane P.
- To get some intuition how coupling is useful here, we will give a short sketch of the proof for the simplest case.

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- For a subset $Q \subset \mathbb{R}^d$, let $\tau_Q = \inf\{t > 0 : B_t \notin Q\}$. The exit time for B_t. Similarly define $\tau'_Q = \inf\{t > 0 : B'_t \notin Q\}$

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- For $x \in D \subset \mathbb{R}^d$, define $\delta_x = \text{dist}(x, D^c)$ and for a function on Q, let

$$\text{osc}_Q u \equiv \sup_Q u - \inf_Q u.$$

Cranston's Gradient Estimate

Theorem (Cranston, 92)

Let u solve $\frac{1}{2}\Delta u = f$ *on* $D \subset \mathbb{R}^d$. *Then with*

$$Q = \left\{ y \in \mathbb{R}^d : \max_{1 \le i \le d} \left| y_i - \frac{x_i - y_i}{2} \right| \le \frac{\delta_x}{2} \right\},$$

we have

$$|\nabla u(x)| \le \frac{d}{\delta_x} osc_Q u + \frac{1}{4} \delta_x osc_Q f.$$

Assume b is a vector field on D with $|b(x)| \le m$ *and* $|b(x) - b(y)| \le m|x - y|$ *for all* x, y. *Let u solve* $\frac{1}{2}\Delta u(x) + b(x) \cdot \nabla u(x) = 0$, $x \in D$ *then there is a constant* $c = c(m)$ *such that*

$$|\nabla u(x)| \le 2\left(1 + \frac{c}{\delta_x}\right) osc_Q u.$$

Sketch of Proof for Poisson's Equation

- Let $x, y \in D \subset \mathbb{R}^d$ with $|x-y| < \frac{\delta_x}{4}$. Note that $\tau_Q = \tau'_Q$.

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- Let $x, y \in D \subset \mathbb{R}^d$ with $|x-y| < \frac{\delta_x}{4}$. Note that $\tau_Q = \tau_Q'$.
- Using the Lindvall-Roger's coupling and Ito's formula one gets

$$
\begin{aligned}
u(B_{\tau_Q}) - u(x) &= \int_0^{\tau_Q} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} (B_s)\, dB_s^{(i)} + \frac{1}{2} \int_0^{\tau_Q} (\Delta u\,(B_s))\, ds \\
u(B'_{\tau_Q}) - u(y) &= \int_0^{\tau_Q} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \left(B'_s\right) dB_s'^{(i)} + \frac{1}{2} \int_0^{\tau_Q} \left(\Delta u \left(B'_s\right)\right) ds
\end{aligned}
$$

so that

$$
\begin{aligned}
u(x) - u(y) &= u(B_{\tau_Q}) - u(B'_{\tau_Q}) \\
&\quad + \int_0^{\tau_Q} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \left(B'_s\right) dB_s'^{(i)} - \int_0^{\tau_Q} \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} (B_s)\, dB_s^{(i)} \\
&\quad - \frac{1}{2} \int_0^{\tau_Q} \left(\Delta u\,(B_s) - \Delta u \left(B'_s\right)\right) ds.
\end{aligned}
$$

Sketch of Proof

Taking expectations and using the fact that stochastic integrals are martingales hence they have zero expectation, we have

$$\begin{aligned} u(x)-u(y) &= \mathbb{E}\left[u(B_{\tau_Q})-u(B'_{\tau_Q})\right]-\frac{1}{2}\mathbb{E}\int_0^{\tau_Q}\left(\Delta u\left(B_s\right)-\Delta u\left(B'_s\right)\right)ds, \\ &= \mathbb{E}\left[u(B_{\tau_Q})-u(B'_{\tau_Q})\right]-\mathbb{E}\int_0^{\tau_Q}\left(f\left(B_s\right)-f\left(B'_s\right)\right)ds, \end{aligned}$$

so that

$$|u(x)-u(y)| \leq \left|\mathbb{E}\left[u(B_{\tau_Q})-u(B'_{\tau_Q})\right]\right| + \left|\mathbb{E}\int_0^{\tau_Q}\left(f\left(B_s\right)-f\left(B'_s\right)\right)ds\right|.$$

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- The first expectation is nonzero since $u(B_{\tau_Q}) - u(B'_{\tau'_Q}) \neq 0$ on $(T > \tau_Q)$. Similarly the second integrand is nonzero on $\{s \mid 0 < s < T \wedge \tau_Q\}$. Thus

$$|u(x) - u(y)| \leq (\text{osc}_Q u)\, P^{x,x'}\,(T > \tau_Q) + (\text{osc}_Q f)\, E^{x,x'}\,[T \wedge \tau_Q]\,.$$

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  - This proves the first gradient estimate for Poisson's equation.

The Kendall Coupling

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- Kendall proved that simply connected manifolds in which all sectional curvatures are negative and bounded away from zero, do not have BCP.

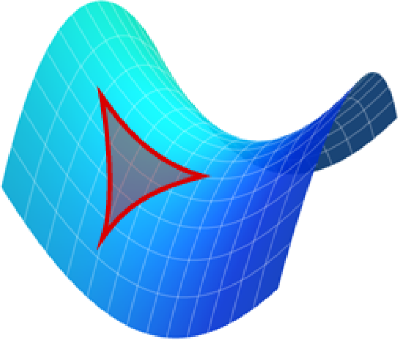
Theorem (Kendall, 1986)

Let M be a simply connected, complete Riemannian manifold of sectional curvature no greater than $-h^2 < 0$. If

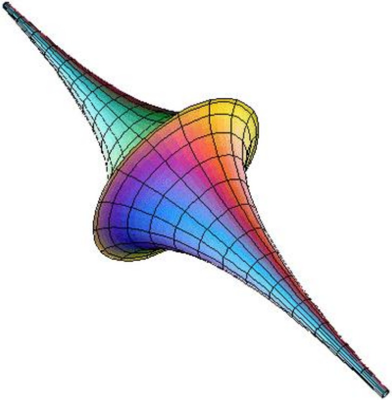
$$F : \mathbb{R}^m \to M$$

is harmonic and of $\kappa-$bounded dilation then F is constant.

Examples of Negative Curvature

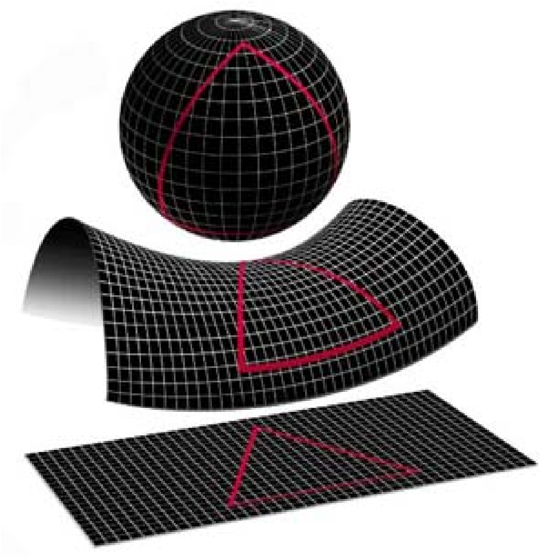


The saddle



The pseudosphere

Curvature



Cheng-Yau's Gradient Estimate

Theorem (Cheng-Yau, 1975)

If M is a complete Riemannian manifold of dimension $d \geq 2$ with $Ric \geq -(d-1)K$ for some $K \geq 0$, then any positive harmonic function u on B_r satisfies

$$\sup_{B_{r/2}} \frac{|\nabla u|}{u} \leq C_d \left(\frac{1}{r} + \sqrt{K} \right)$$

where B_r denotes any geodesic ball of radius r. In particular,

$$\sup_{B_{r/2}} u \leq e^{2C_d\left(1+r\sqrt{K}\right)} \inf_{B_{r/2}} u.$$

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- Similar to the Euclidean case, he applies this coupling to prove a gradient estimate on manifolds with curvature bounded below.

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- Similar to the Euclidean case, he applies this coupling to prove a gradient estimate on manifolds with curvature bounded below.
  - Recall the key ingredient in proving the gradient estimate was to estimate the probability that the coupling is unsuccessful.
- For some open set $Q \subset M$ define $\tau_Q(X) = \inf\{t > 0 : X_t \notin Q\}$, the exit time of the set Q.

Cranston's Main Result on Manifolds

Theorem (Cranston, 1991)

Suppose (M, g) is a complete $d-$dimensional Riemannian manifold with distance ρ_M and $Ric_M \geq -Kg$. Let Z be a C^1 vector field on M such that $|Z(x)| \leq m$ for all $x \in M$ for some $m \geq 0$. Then there is a constant $c = c(K, d, m)$ such that for all $x, y \in B(x_0, \delta)$,

$$\mathbb{P}^{(x,y)}\left(T(X, Y) > \tau_{B(x_0,2\delta)}(X) \wedge \tau_{B(x_0,2\delta)}(Y)\right) \leq c\left(\frac{1}{\delta} + 1\right)\rho_M(x, y).$$

Furthermore,

$$\mathbb{P}^{(x,y)}\left(T(X, Y) = \infty\right) \leq \left(2\sqrt{K(d-1)} + 2m\right)\rho_M(x, y).$$

Consider the First Estimate

Take $Q = B(x_0, 3\delta/2)$. Using the first inequality from Cranston's Theorem we have that for $x, y \in B\left(x_0, \delta\right)$,

$$|u(x) - u(y)| \;\; = \;\; \left|\mathbb{E}^{(x,y)}\left[u\left(X_{\tau_Q}\right) - u\left(Y_{\tau_Q}\right); T\left(X, Y\right) > \tau_Q\left(X\right) \wedge \tau_Q\left(Y\right)\right]\right.$$

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Dividing through $\rho_M(x, y)$ and taking $y \to x$, we get

$$\boxed{|\nabla u(x)| \leq c \left(\frac{1}{\delta} + 1 \right) \sup_{B(x_0, 3\delta/2)} u.}$$

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Assume $u \geq 0$ and bounded on M,

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$$

where we used the second inequality from the Cranston's theorem . Then

$$
|\nabla u(x)| \leq 2 \left\|u\right\|_{\infty} \left(\sqrt{K(d-1)} + m\right).
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Theorem (Cranston, 1991)

Suppose (M, g) and Z are as in the previous theorem. There is a constant $c = c(K, d, m)$ such that whenever $\delta > 0$ and $Lu = 0$ in some $B(x_0, 2\delta)$, we get

$$|\nabla(x)| \leq c\left(\frac{1}{\delta} + 1\right) \sup_{B(x_0, 3\delta/2)} u, \quad x \in B(x_0, \delta).$$

If $Lu = 0$ on M and u is bounded and positive, then

$$|\nabla u(x)| \leq 2\left(\sqrt{K(d-1)} + m\right) \|u\|_\infty.$$

Kendall-Cranston Coupling

- For $x \in M$ and $y \notin C(x)$, the cut locus of x, let $\sigma_C(Y) = \inf\{t > 0 : Y_t \in C(X_t)\}$ be the first time Y_t hits the cut locus of X_t. Which we will refer to the hitting time of the cut locus.

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- He also makes use of the second variation of arc length.

Kendall-Cranston Coupling

Theorem (Cranston, 1991)

The process Y may be continued past $\sigma_C(Y)$ in such a way that Y is a diffusion on M with generator $\frac{1}{2}\Delta + Z$. Furthermore, there is an increasing process L which increases only on the set of times $\{t : Y_t \in C(X_t)\}$ such that for all $t < T(X, Y)$

$$d\rho_M(X_t, Y_t) = 2b_t + \left[\int_{X_t}^{Y_t} \sum_{i=2}^{d} \left(\left|\nabla_T W^i\right|^2 - \left\langle R\left(W^i, T\right) T, W^i \right\rangle \right) dt \right] + \left[\langle Z(Y_t), T\rangle - \langle Z(X_t), T\rangle\right] dt - dL_t.$$

The coefficients of dt are set equal to zero on the support of L.

Proof of the Main Theorem

- The proof of Cranston's main theorem depends on an estimate of the index form

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- Also a estimate on

$$d\rho_M(X_t, x) = dw_t + \left(\frac{1}{2}\Delta + Z\right) \rho_M(X_t, x)dt - dL_t^x$$

  is made using the Laplacian Comparison Theorem.

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  - The left-invariant vector fields are

$$\begin{aligned} X &= \partial_x - \frac{y}{2}\partial_z \\ Y &= \partial_y + \frac{x}{2}\partial_z \\ Z &= \partial_z. \end{aligned}$$

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- Here we lose all comparison techniques available to us, such as the Index Theorem, the Laplacian Comparison Theorem and so on.
- One idea in using the Kendall-Cranston coupling is to approximate the sub-Riemannian manifold with Riemannian manifolds.
- So one could consider a collection of "Riemannian" Heisenberg groups $\mathbb{H}_\epsilon$ with orthonormal basis $\{X, Y, \epsilon Z\}$ such that the Riemannian $\mathbb{H}^3_\epsilon$ "converges" to the sub-Riemannian $\mathbb{H}^3$.

Thank you!