The volume of the unit ball in n dimensions

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Outline

Goal

- The unit ball in 1D is a line segment of length 2. In 2D, it is the unit disk with area $\pi \cong 3.14$. In 3D, it is the ball with volume $\frac{4}{3}\pi \cong 4.188$.
- There is no nice way to visualize the n-ball for $n > 3$. What is the volume of the n-ball? Does it keep growing as n approaches infinity?
- We will find an explicit formula for the volume of the n-ball and observe a behavior that seems to go against our 3-dimensional intuition.

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Definitions

- Let $B_n(R)$ to denote a solid ball of radius R, centered at the origin which lives in $\mathbb{R}^n$. Namely

$$B_n(R) = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\right\},$$

- Let

$$V_n = Vol(B_n(1))$$

be the volume of the unit ball. We can write the volume of the ball of radius R in terms of V_n.

$Vol\left(B_n\left(R\right)\right) = V_n R^n.$

Theorem

$Vol\left(B_n\left(R\right)\right) = V_n R^n.$

- Compute:

$$\begin{aligned} B_n\left(R\right) &= \left\{ \left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq R^2 \right\} \\ &= \left\{ R\left(\frac{x_1}{R}, \frac{x_2}{R}, \ldots, \frac{x_n}{R}\right) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \left(\frac{x_i}{R}\right)^2 \leq 1 \right\} \\ &= RB_n\left(1\right). \end{aligned}$$

- Then $Vol\left(B_n\left(R\right)\right) = Vol\left(RB_n\left(1\right)\right) = R^n B_n\left(1\right) = R^n V_n$, by dilation in n-dimensions.

Examples

Example

In $\mathbb{R}^5$ we have

$$B_5(1) = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 1\right\}.$$

The point $\left(\frac{1}{10}, 0, \frac{1}{5}, 0, \frac{1}{2}\right) \in B_5(1)$ since

$$\left(\frac{1}{10}\right)^2 + 0^2 + \left(\frac{1}{5}\right)^2 + 0^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{10} \leq 1.$$

Example

Define $H = \left\{(x_1, x_2, x_3, x_4, 0) \in \mathbb{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\right\}$. Then $B_5(1)$ contains a copy of $B_4(1)$ inside it.

Outline

Calculation of V_3

- We add up the circular slices of radius $\cos\theta_i$ with thickness $d(\sin\theta_i)$ with $\theta_i's$ varying from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
- Each slice has area $A_i = \pi(\cos\theta_i)^2 = \pi\cos^2\theta$ and volume $Vol_i = A_i \times d(\sin\theta_i) = \pi\cos^2\theta_i d(\sin\theta_i)$
- We get that

$$
\begin{aligned}
V_3 &= \lim_{n\to\infty}\sum_{i=1}^{n}\pi\cos^2\theta d(\sin\theta_i) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi\cos^2\theta d(\sin\theta) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi\cos^3\theta d\theta = 2V_2\int_0^{\frac{\pi}{2}}\cos^3\theta d\theta.
\end{aligned}
$$

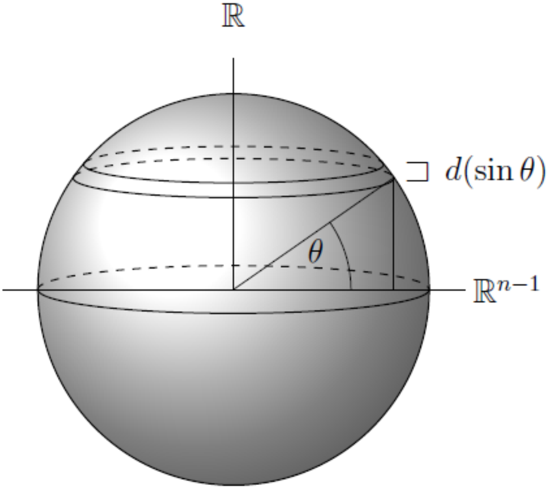
Calculation of V_n

- **Goal:** To show that the pattern continues so that

$$V_n = 2V_{n-1}\int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta.$$

To show this formula, we repeat the same argument done in the $n = 3$ case.

Calculation of V_n



Calculation of V_n

We partition the x_n-axis into $\sin\theta_1,\ldots,\sin\theta_n$. Let H_i be the set of all points of the form $(x_1,\ldots,x_{n-1},\sin\theta_i)\in B_n(1)$, which is represented as a horizontal slice. We claim H_i is copy of the $(n-1)$ dimensional ball of radius $\cos\theta_i$. We have,

$$\begin{aligned}(x_1,\ldots,x_{n-1},\sin\theta_i)\in H_i &\iff x_1^2+x_2^2+\cdots+x_{n-1}^2+\sin^2\theta_i\leq 1\\ &\iff x_1^2+x_2^2+\cdots+x_{n-1}^2\leq 1-\sin^2\theta_i\\ &\iff x_1^2+x_2^2+\cdots+x_{n-1}^2\leq \cos^2\theta_i\\ &\iff (x_1,\ldots,x_{n-1})\in B_{n-1}(\cos\theta_i).\end{aligned}$$

Calculation of V_n

So now each wedge has volume

$$\begin{aligned} Vol_i &= Vol\left(B_{n-1}\left(\cos\theta_i\right)\right) d\left(\sin\theta_i\right) \\ &= V_{n-1}\cos^{n-1}\theta_i d\left(\sin\theta_i\right), \end{aligned}$$

by a previous theorem. Thus

$$\begin{aligned} V_n &= \lim_{n\to\infty}\sum_{i=1}^{n} V_{n-1}\cos^{n-1}\theta_i d\left(\sin\theta_i\right) \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V_{n-1}\cos^{n-1}\theta d\left(\sin\theta\right) \\ &= 2V_{n-1}\int_0^{\frac{\pi}{2}}\cos^{n}\theta d\theta. \end{aligned}$$

Outline

Definition of I_n

Since $V_n = 2V_{n-1}\int_0^{\frac{\pi}{2}} \cos^n\theta d\theta$ define

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n\theta d\theta,$$

so that

$$V_n = 2V_{n-1}I_n. \tag{1}$$

We prove the following recursive formula for I_n:

Theorem

$I_n = \frac{n-1}{n} I_{n-2}$ *for all n.*

Proof of $I_n = \frac{n-1}{n} I_{n-2}$

Using integration by parts with

$$\begin{aligned} u &= \cos^{n-1}\theta & dv &= \cos\theta\, d\theta \\ du &= (n-1)\cos^{n-2}\theta\,(-\sin\theta) & v &= \sin\theta, \end{aligned}$$

so that

$$\begin{aligned} I_n &= \left[\cos^{n-1}\theta \sin\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \sin^2\theta\, d\theta \\ &= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \left(1 - \cos^2\theta\right) d\theta \\ &= (n-1)\left[\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta\, d\theta - \int_0^{\frac{\pi}{2}} \cos^{n}\theta\, d\theta\right] = (n-1)\left[I_{n-2} - I_n\right]. \end{aligned}$$

Formula for I_n.

Let's try to find a pattern.

$$I_0 = \int_0^{\frac{\pi}{2}} (\cos\theta)^0 \, d\theta = \frac{\pi}{2}$$
$$I_1 = \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \sin\theta\Big|_0^{\frac{\pi}{2}} = 1.$$

Then we use the recursive formula we just found $\left(I_n = \frac{n-1}{n} I_{n-2}\right)$, to find the rest. Namely

$$I_2 = \frac{1}{2} I_0 = \frac{1}{2}\frac{\pi}{2} \qquad I_3 = \frac{2}{3} I_1 = \frac{2}{3}$$
$$I_4 = \frac{3}{4} I_2 = \frac{3}{4} \qquad I_5 = \frac{4}{5} I_3 = \frac{2}{3}\frac{4}{5}$$

Formula for I_n.

Continuing on we get

$$\begin{aligned} I_2 &= \frac{1}{2} I_0 = \frac{1}{2}\frac{\pi}{2} \\ I_4 &= \frac{3}{4} I_2 = \frac{1}{2}\frac{3}{4}\frac{\pi}{2} \\ I_6 &= \frac{5}{6} I_4 = \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{\pi}{2} \\ I_8 &= \frac{7}{8} I_2 = \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\frac{\pi}{2} \\ I_{10} &= \frac{9}{10} I_2 = \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\frac{9}{10}\frac{\pi}{2} \\ &\vdots (n = \text{even}) \\ I_n &= \frac{\pi}{2}\frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots \cdot n} \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{2}{3} I_1 = \frac{2}{3} \\ I_5 &= \frac{4}{5} I_3 = \frac{2}{3}\frac{4}{5} \\ I_7 &= \frac{6}{7} I_5 = \frac{2}{3}\frac{4}{5}\frac{6}{7} \\ I_9 &= \frac{8}{9} I_3 = \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9} \\ I_{11} &= \frac{10}{11} I_3 = \frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\frac{10}{11} \\ &\vdots (n = \text{odd}) \\ I_n &= \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n}. \end{aligned}$$

Now what happens when you multiply I_6, I_7 or I_{10}, I_{11} together?

$I_n I_{n-1} = \frac{\pi}{2n}$.

Theorem

For any $n > 0$ we have $I_n I_{n-1} = \frac{\pi}{2n}$.

The base case $n = 1$ is simple since $I_0 = \frac{\pi}{2}, I_1 = 1$ and clearly

$$I_1 I_0 = \frac{\pi}{2 \cdot 1}.$$

Now assume the formula is true for integers less than n. We prove it for n. From $I_n = \frac{n-1}{n} I_{n-2}$ we have

$$\begin{aligned} I_n = \frac{n-1}{n} I_{n-2} &\iff n I_n = (n-1) I_{n-2} \\ &\iff n I_n I_{n-1} = (n-1) I_{n-2} I_{n-1}. \end{aligned}$$

$I_n I_{n-1} = \frac{\pi}{2n}$.

Using this we get that

$$\begin{aligned} nI_n I_{n-1} &= (n-1)\, I_{n-2} I_{n-1} \\ &= (n-1)\,\frac{\pi}{2\,(n-1)} \text{ by inductive hypothesis} \\ &= \frac{\pi}{2}. \end{aligned}$$

Dividing by n we arrive at

$$I_n I_{n-1} = \frac{\pi}{2n}. \tag{2}$$

Outline

Formula for V_n

We want to find an explicit formula for V_n using equations (1) and (2), namely

$$V_n = 2I_n V_{n-1} \text{ and } I_n I_{n-1} = \frac{\pi}{2n}.$$

Replace n with $n-1$ and get

$$V_{n-1} = 2I_{n-1} V_{n-2}.$$

$V_n = \frac{2\pi}{n} V_{n-2}$

Plugging this into equation (1) and get

$$\begin{aligned} V_n &= 2I_n V_{n-1} \\ &= 2I_n \left[2I_{n-1} V_{n-2}\right] \\ &= 4\left[I_n I_{n-1}\right] V_{n-2} \\ &= 4\frac{\pi}{2n} V_{n-2} \\ &= \frac{2\pi}{n} V_{n-2}. \end{aligned}$$

so that $V_n = \frac{2\pi}{n} V_{n-2}$.

Formula for V_n

We search for a pattern. Using our known values of $V_1 = 2, V_2 = \pi, V_3 = \frac{4}{3}\pi$ and the formula $V_n = \frac{2\pi}{n} V_{n-2}$ we get

$$V_4 = \frac{2\pi}{4} V_2 = \frac{\pi^2}{1 \cdot 2} \qquad V_5 = \frac{2\pi}{5} V_3 = 2 \cdot \frac{2^2 \pi^2}{1 \cdot 3 \cdot 5}$$
$$V_6 = \frac{2\pi}{6} V_4 = \frac{\pi^3}{1 \cdot 2 \cdot 3} \qquad V_7 = \frac{2\pi}{7} V_5 = 2 \cdot \frac{2^3 \pi^3}{1 \cdot 3 \cdot 5 \cdot 7}$$
$$V_8 = \frac{2\pi}{8} V_6 = \frac{\pi^4}{1 \cdot 2 \cdot 3 \cdot 4} \qquad V_9 = \frac{2\pi}{9} V_7 = 2 \cdot \frac{2^4 \pi^4}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}.$$

$$V_n = \begin{cases} 2 \cdot \frac{(2\pi)^{\frac{n-1}{2}}}{3 \cdot 5 \cdot 7 \cdots n} & n = \text{odd} \\ \frac{\pi^{\left(\frac{n}{2}\right)}}{\left(\frac{n}{2}\right)!} & n = \text{even} \end{cases}. \tag{3}$$

Outline

What is $\lim_{n\to\infty} Vol(B_n(R))$?

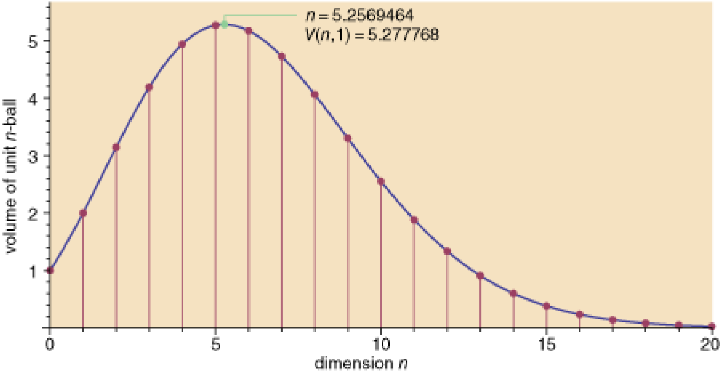
Let's look at some numerical values for small n.

n	V_n	n	V_n
1	2	6	5.17
2	3.14	7	4.72
3	4.19	8	4.06
4	4.93	9	3.30
5	5.26	10	2.55

Any guesses on the limit of V_n as $n \to \infty$?

Graph of behavior

Plotting the dimension n in the $x-$axis and the value of V_n in the $y-$axis we see this surprising behavior. The Volume peaks at $n = 5$:



What is $\lim_{n\to\infty} Vol(B_n(R))$?

Theorem

For $R > 0$ we have that

$$\lim_{n\to\infty} Vol(B_n(R)) = 0.$$

Take $n = 2k$ even. Then by (3) we have that $Vol(B_{2k}(R)) = \frac{(R\pi)^k}{k!}$.
Recall *Stirling's formula (again!)* :

$$k! \sim \sqrt{2\pi k}\, k^k e^{-k}.$$

What is $\lim_{n\to\infty} Vol(B_n(R))$?

Plugging Stirling's Formula into $Vol(B_{2k}(R))$ we easily see that

$$\begin{aligned} Vol(B_{2k}(R)) &= \frac{\left(R^2\pi\right)^k}{k!} \\ &\sim \frac{\left(R^2\pi\right)^k}{\sqrt{2\pi k}\, k^k e^{-k}} \\ &\sim \frac{1}{\sqrt{2\pi k}} \cdot \left(\frac{C}{k}\right)^k, \end{aligned}$$

where $C = R^2\pi e > 0$. From calculus this approches zero as $k \to \infty$.

What is $\lim_{n\to\infty} Vol(B_n(R))$?

When n is odd the formula is a bit more tricky. We have that $Vol(B_{2k+1}(R)) = 2 \cdot \frac{(2\pi)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)}$. The product in the denominator is sometimes called a double factorial, written as $n!!$. It is not too hard to see that

$$(2k)!! = 2 \cdot 4 \cdot 6 \cdots (2k) = 2^k k!.$$

Without proof we have that

$$(2k-1)!! = \frac{(2k)!}{2^k k!}.$$

What is $\lim_{n\to\infty} Vol(B_n(R))$?

Take $n = 2k - 1$ odd and using (3) we get

$$\begin{aligned} Vol(B_{2k-1}(R)) &= 2 \cdot \frac{(2\pi)^{k-1}}{(2k-1)!!} \\ &= 2 \cdot \frac{(2\pi)^{k-1} 2^k k!}{(2k)!} \\ &\sim 2 \cdot \frac{(2\pi)^{k-1} 2^k \sqrt{2\pi k} k^k e^{-k}}{\sqrt{2\pi 2k} (2k)^{2k} e^{-2k}} \text{ by Sterling's Formula} \\ &\sim \frac{1}{\pi\sqrt{2}} \left(\frac{\pi e}{k}\right)^k \end{aligned}$$

which approches zero as $k \to \infty$.

Explanation for behavior

- You can't compare the measure of Volume in $2D$ with $3D$, or $n - dim$ since the units are not the same.
- Hypersheres are a bad approximation for the hypercube in high dimensions.
  - They coincide in dimension 1, it gets worse from there.
- In some sense, in high dimensional space, most of the volume of the unit ball is concetrated near the surface away from the center.

Concentration near the surface

- Why is most of the volume concentrated near the surface? We can show that almost all the volume of $B_n(1)$ is concentrated near any spherical shell for large n.

Consider $B_n(1), B_n(\varepsilon)$ (concentric balls) where $0 < \varepsilon < 1$. Then

$$\frac{Vol\left(B_n\left(\varepsilon\right)\right)}{Vol\left(B_n\left(1\right)\right)} = \frac{V_n \varepsilon^n}{V_n} = \varepsilon^n \to 0 \text{ as } n \to \infty.$$

But this is the fraction of of the volume contained in the center, which approaches zero. Thus most of the volume is near the surface as $n \to \infty$.

Concentration near Equator

Let the equator be defined as $E = \{(x_1, \ldots, x_n) \in B_n(1) \mid x_n = 0\}$. Let H_ε be the upper hemishere above the plane $x_n = \varepsilon$, namely $H_\varepsilon = \{(x_1, \ldots, x_n) \in B_n(1) \mid \varepsilon \le x_n \le 1\}$. As before we have that

$$Vol(H_\varepsilon) = V_{n-1} \int_{\sin^{-1}\varepsilon}^{\frac{\pi}{2}} \cos^{n-1}\theta \, d(\sin\theta).$$

To make an estimate we switch back to rectangular coordinates so that

$$Vol(H_\varepsilon) = V_{n-1} \int_{\varepsilon}^{1} \left(1 - x_n^2\right)^{\frac{n-1}{2}} dx_n.$$

Concentration near Equator

We recall that $1+x \leq e^x$ on $\mathbb{R}$ and use the fact that $\varepsilon \leq x_n$ so that $1 \leq \frac{x_n}{\varepsilon}$ and

$$Vol(H_\varepsilon) = V_{n-1} \int_\varepsilon^1 \left(1 - x_n^2\right)^{\frac{n-1}{2}} dx_n \leq V_{n-1} \int_\varepsilon^\infty x_n e^{\frac{-x_n^2(n-1)}{2}} dx_n.$$

We can calculate this integral:

$$Vol(H_\varepsilon) \leq \frac{V_{n-1}}{\varepsilon(n-1)\exp\left(-\frac{n-1}{2}\varepsilon^2\right)}.$$

Concentration near Equator

We now find a lower bound for $Vol\left(H_0\right)$, which is the volume of the entire hemisphere.

- This volume is bounded below by the volume between the planes $x_n = 0$ and $x_n = \frac{1}{\sqrt{n-1}}$.
- Hence it should be bounded by the cylinder of radius $r = \sqrt{1 - x_1^2} = \sqrt{1 - \frac{1}{n-1}}$ with height $h = \frac{1}{\sqrt{n-1}}$. By our first Theorem, this cylinder has volume

$$V\left(\sqrt{1 - \frac{1}{n-1}}\right)\frac{1}{\sqrt{n-1}} = \left[V_{n-1}\left(\sqrt{1 - \frac{1}{n-1}}\right)^{n-1}\right]\frac{1}{\sqrt{n-1}}.$$

Concentration near Equator

- Since we said $Vol(H_0)$ is bounded below by this cylinder then

$$Vol(H_0) \geq \left[V_{n-1} \left(1 - \frac{1}{n-1} \right)^{\frac{n-1}{2}} \right] \frac{1}{\sqrt{n-1}}$$

- Using the formula $(1-x)^p \geq 1 - px$ we have

$$Vol(H_0) \geq V_{n-1} \left(1 - \frac{n-1}{2} \frac{1}{n-1} \right) \frac{1}{\sqrt{n-1}} = \frac{V_{n-1}}{2} \frac{1}{\sqrt{n-1}}.$$

Concentration near Equator

Finally we have the following claim:

Theorem

For $c > 0$, the fraction of the volume of the upper hemisphere H_0 that is above the plane $x_n = \frac{c}{\sqrt{n-1}}$, is less than $\frac{2}{c\exp(c^2/2)}$.

To prove this we combine the estimates

$$Vol(H_0) \geq \frac{V_{n-1}}{2}\frac{1}{\sqrt{n-1}} \quad \text{and} \quad Vol(H_\varepsilon) \leq \frac{V_{n-1}}{\varepsilon(n-1)\exp\left(-\frac{n-1}{2}\varepsilon^2\right)}$$

with $\varepsilon = \frac{c}{\sqrt{n-1}}$.

Concentration near Equator

We get that

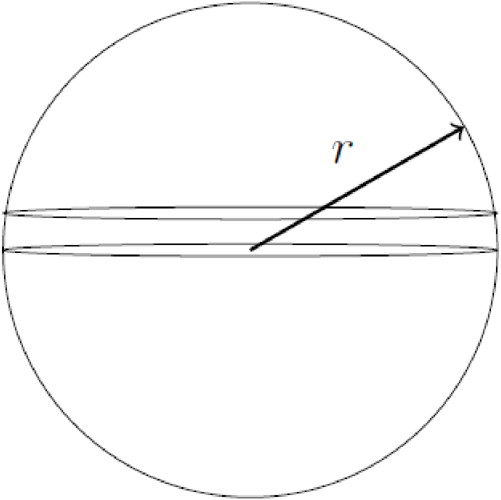
$$\frac{Vol\left(H_{\frac{c}{\sqrt{n-1}}}\right)}{Vol\left(H_0\right)} \leq \frac{V_{n-1}}{\frac{c}{\sqrt{n-1}}\left(n-1\right)\exp\left(-\frac{n-1}{2}\varepsilon^2\right)}\frac{2\sqrt{n-1}}{V_{n-1}}$$
$$= \frac{2}{c\exp\left(c^2/2\right)}.$$

Thus for n and c big enough, we have that $\frac{2}{c\exp(c^2/2)}$ is small, hence most of the volume is located near the equator.

Concentration near Equator- Example

- Take $c = 2$ so that $\frac{2}{c\exp(c^2/2)} = \frac{1}{e^2} \approx 0.14$. So that if the dimension is $n = 1000$, then about 14% of the volume is contained in the slice above the plane $x_n = \frac{c}{\sqrt{n-1}} = \frac{2}{\sqrt{999}} \approx .06$.
- Take $c = 4$ so that $\frac{2}{c\exp(c^2/2)} = \frac{1}{2e^8} \approx .0016$. So that if the dimension is $n = 1000$, then about .16% of the volume is contained in the slice above the plane $x_n = \frac{c}{\sqrt{n-1}} = \frac{4}{\sqrt{999}} \approx .12$.
- Almost all the volume is close to the equator! This works for any equator too!

Graph of H_ε



Summary

- We found

$$V_n = \begin{cases} 2 \cdot \frac{(2\pi)^{\frac{n-1}{2}}}{3 \cdot 5 \cdot 7 \cdots n} & n = \text{odd} \\ \frac{\pi^{\left(\frac{n}{2}\right)}}{\left(\frac{n}{2}\right)!} & n = \text{even} \end{cases}$$

- Our intution in small dimension does not always translate to facts in n-dimensions.
- The $n-$ball has volume close to zero as n is big enough. This happens very quickly.
- Striking result: Most of the volume is located inside the shells near the surface, yet most of the volume is also near the equator.

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