On the Coarse Geometry of L^p: A Coarse Equivalence

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2013 Joint Mathematics Meeting

Outline

1. Why Coarse Geometry?
   - Motivation
2. A Coarse Equivalence to $\mathscr{R}^1$
   - The partition space $\mathscr{P}([a,b])$.
   - Preliminaries
   - Main Result
3. Coarse Geometry of L^p

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   - The partition space $\mathscr{P}([a,b])$.
   - Preliminaries
   - Main Result
3. Coarse Geometry of L^p

Introduction

- The idea behind coarse geometry is to look at spaces only at large scale, to neglect all local, infinitesimal structure (topological or geometric) and consider only the geometric properties that are global.
- If (X, d) is a metrix space, then

$$d'(x,y) = \min\{1, d(x,y)\}$$

induces same topology. In Coarse geometry,

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- **Coarse equivalence** is the analogous form of a homeomorphism, where coarse properties are invariant.
- Why? Utility in work towards Baum-Connes Conjecture and Novikov conjecture.

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Goal

- We first describe some coarse invariant properties.
- We will show that the topological property of separability is not invariant under coarse equivalence.
- We will show that a certain nonseparable space is coarse equivalent to the separable space L^p if and only if $p = 1$.

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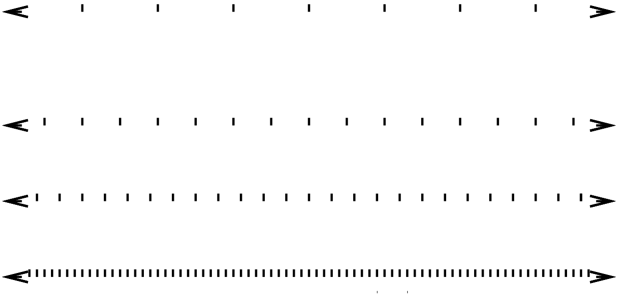
Coarse Equivalence

- Intuitively, X is coarse equivalent to Y if you zoom out far enough on a space X, there will be a point at which X starts to look similar to Y.

Example

The space$(\mathbb{R}, |\cdot|)$ is coarse equivalent to $(\mathbb{Z}, |\cdot|)$.



Definition of a Coarse Function

Definitions

Let X and Y be metric spaces, and let $f : X \to Y$ be any map.
(a) The map f is (metrically) *proper* if the inverse image, under f, of each bounded subset of Y, is a bounded subset of X.
(b) The map f is (uniformly) *bornologous* if for every $R > 0$ there is $S > 0$ such that

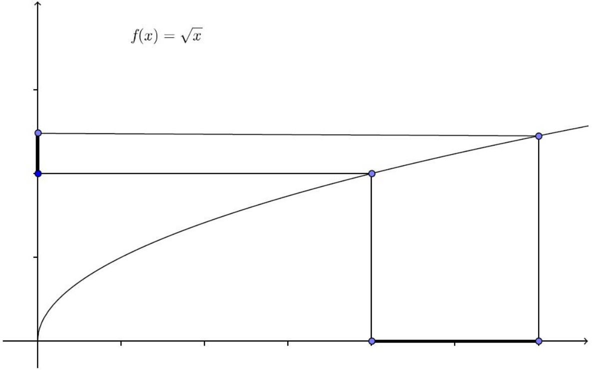
$$d_X(x,y) < R \Rightarrow d_Y(f(x), f(y)) < S$$

(c) The map f is *coarse* if it is proper and bornologous.

Example of a Coarse Function

Example

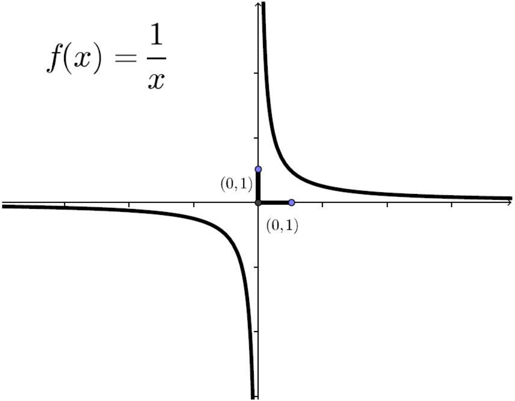
The function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ defined by $f(x) = \sqrt{x}$ is coarse.



Example of a function that is not coarse

Example

The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is *not* coarse.



Definition of a Coarse Equivalence

Definitions

(a) Two maps f, f' from a set X into a metric space Y is *close* if $d\,(f(x), f'(x))$ is bounded uniformly in X.
(b) We say metric spaces X and Y are *coarse equivalent* if there exists coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on Y and on X, respectively.

Examples

We show $(\mathbb{R}, |\cdot|)$ is coarse equivalent to $(\mathbb{Z}, |\cdot|)$. Define $F : \mathbb{R} \to \mathbb{Z}$ by $F(x) = \lfloor x \rfloor$ and define $G : \mathbb{Z} \to \mathbb{R}$ by $G(x) = x$. Its easy to see that F, G are coarse. Note that

$$\begin{aligned} |F \circ G(x) - x| &= |\lfloor x \rfloor - x| \\ &= |x - x| \\ &\leq C, \quad \forall C > 0 \end{aligned}$$

and

$$\begin{aligned} |G \circ F(x) - x| &= |\lfloor x \rfloor - x| \\ &\leq 1 \end{aligned}$$

and so $(\mathbb{R}, d)$ and $(\mathbb{Z}, d)$ are coarse equivalent.

Examples: Groups as Metric Spaces

- Interesting examples involve defining metrics on finitely generated groups.
- Let G be a group and suppose Γ generates G; that is every $g \in G$ can be written as a *word* of members of Γ and their inverses; that is

$$g = \gamma_1 \gamma_2 \cdots \gamma_n$$

  and we let the smallest number n of generators that can be used to form g to be called the **word length** (relative to Γ), and write it as$|g|$.
- It is not too hard to show that $d(g,h) = |g^{-1}h|$ defines a metric on G, known as the **word metric**.

Examples: Groups as Metric Spaces

Theorem

Let Γ and Γ' be two generating sets for the same group G; and let d and d' be the associated word metrics. Then the identity map $(G,d) \to (G,d')$ is a coarse equivalence.

- Thus any finitely generated group carries an intrinsic coarse geometry.

Invariant Coarse Geometric Properties

- Bounded Geometry: $\forall R > 0, \exists N(r) > 0$ such that $\#(B(x,r)) < N(r)$ for all x.
- Asymptotic dimension: Similar to Covering Dimension.
- Amenability:
- Property A

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Asymptotic Dimension

- Coarse version of the Lebesgue Covering Dimension of Topology

Definition

(**Gromov**) The asymptotic dimension of a metric space X is the smallest integer n such that for every $R > 0$, there exists a uniformly bounded cover $\mathscr{U} = \{U_i\}_{i \in I}$ of X so that
(1) The diameters of $U_i's$ are bounded,
(2) Each $R - ball$ intersects at most $n+1$ members of the covering
We write

$$\text{asdim } X = n.$$

Calculating asdim $\mathbb{Z}$

To calculate asdim $\mathbb{Z}$ we let $R > 0$ and consider collection

$$\mathscr{U} = \{[2Rn, 2(n+1)R)\}_{n\in\mathbb{Z}}$$

Taking $R = 1$ we have

$$\mathscr{U} = \{\ldots, [-4,-2), [-2,0), [0,2), [2,4)\}.$$

Note that these sets are disjoint, with length $2R$ and hence uniformly bounded. Also note that a ball of radius R centered at any point $x \in \mathbb{Z}$ can intersect at most 2 elements of $\mathscr{U}$.
This shows

$$\text{asdim } \mathbb{Z} \leq 1.$$

We can also show that asdim $\mathbb{Z} \geq 1$, hence asdim $\mathbb{Z} = 1$.

- Since $\mathbb{Z}$ is coarse equivalent to $\mathbb{R}$, then asdim $\mathbb{R} = 1$.

Why Coarse Geometry?

Theorem

*(**Yu**) Let G be a finitely generated group whose classifying space BG has the homotopy type of a finite CW-complex. If G has finite asymptotic dimension as a metric space with a word length metric, then the Novikov conjecture holds for G.*

Theorem

*(**Yu**)Let Γ be a discrete metric space with bounded geometry. If Γ admits a uniform embedding into the Hilbert Space, then the Baum-Connes conjecture holds for Γ.*

Outline

Goal

- Using function spaces over $[a,b]$, we show Separability is not Coarse Invariant.
- We first construct the Nonseparable space $\mathscr{P}([a,b])$.

Definitions

Definition

A *partition* P of $[a,b]$ is an ordered set $P = \{a = x_0, x_1, \ldots x_{n-1}, x_n = b\}$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

Definition

A *partition function* f_P of an interval $[a,b]$, is a function $f_P : P \to \mathbb{R}$ where P is a partition of $[a,b]$.

- We set the **Partition Space** $\mathscr{P}([a,b])$ to be the set of all partition functions $f_P : P \to \mathbb{R}$ on $[a,b]$.

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The function *Lin*

- We define the the **linear interpolating operator.**

Definition

Let f_P be a partition function on $[a,b]$. Define $Lin : \mathscr{P}([a,b]) \to \mathscr{R}^1$ by

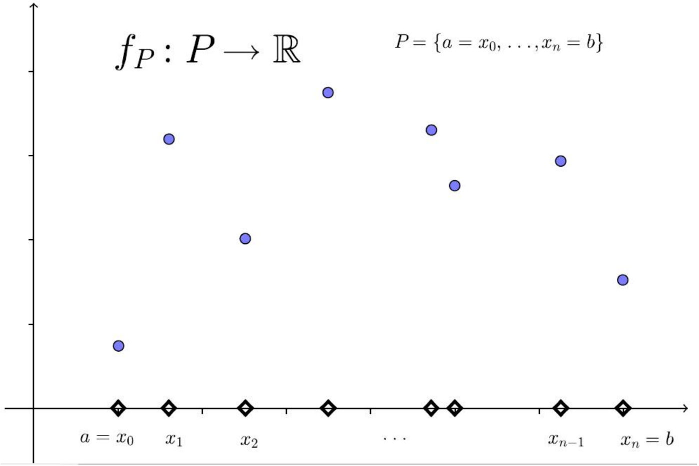
$$Lin(f_P(x)) := \begin{cases} f_P(x_i) & ,x = x_i \\ \frac{f_P(x_i)-f_P(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}) + f_P(x_{i-1}) & ,x \in (x_{i-1}, x_i) \end{cases}$$

Theorem

The function $Lin(f_P)$ is coarse.

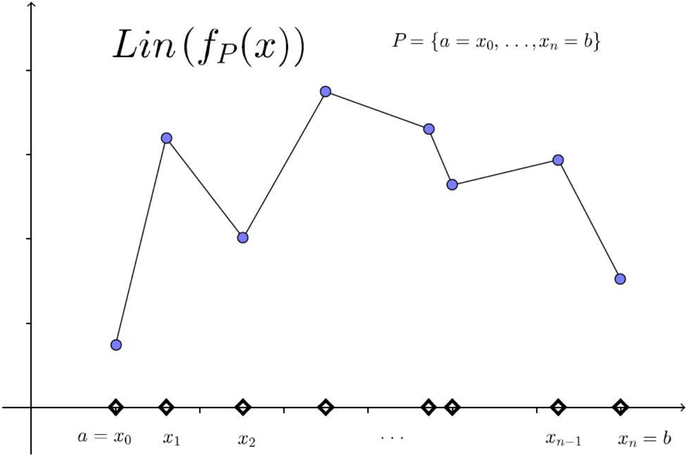
A Partition function

- The partition function f_P



The *Lin* function

- The function $Lin(f_P)$.



The metric space $\mathscr{P}([a,b])$

Theorem

Define $d_{\mathscr{P}} : \mathscr{P}([a,b]) \times \mathscr{P}([a,b]) \to \mathbb{R}$ *for* $f_{P_1}, g_{P_2} \in \mathscr{P}([a,b])$ *as*

$$d_{\mathscr{P}}(f_{P_1}, g_{P_2}) = \begin{cases} \int_a^b |Lin(f_{P_1}(x)) - Lin(g_{P_2}(x))|\, dx & , P_1 = P_2 \\ \int_a^b |Lin(f_{P_1}(x)) - Lin(g_{P_2}(x))|\, dx + 1 & , P_1 \neq P_2 \end{cases}$$

then $(\mathscr{P}([a,b]), d_{\mathscr{P}})$ *is a metric space.*

Separability

Definition

A metric space X is said to be separable if it contains a countable dense subset.

Theorem

$\mathscr{P}([a,b])$ is not separable.

- Define the partition function $g_P(x) = 1$ where P is a partition, and set $A = \{g_P \mid P \text{ is a partition of } [a,b]\}$. We can find a 1-1 map $F : A \to \mathbb{N}$; a contradiction since A is uncountable.

Outline

- We define $\mathscr{R}^1([a,b])$ to be the set of Riemann Integrable functions on $[a,b]$.
- We recall from analysis that for a partition P of $[a,b]$ and function $f:[a,b]\to\mathbb{R}$ we have
- $$U(f,P)=\sum_{i=1}^{n} M_i\Delta x_i \qquad \text{and} \qquad L(f,P)=\sum_{i=0}^{n} m_i\Delta x_i.$$

  where M_i, m_i are the sup's and inf's of respective intervals.

Theorem

$\mathscr{R}^1([a,b])$ is separable.

Outline

Main Result

- We prove that there exists a nonseparable space coarse equivalent to a separable space.

Theorem

$(\mathscr{P}([a,b]), d_{\mathscr{P}})$ is coarse equivalent to $\left(\mathscr{R}^1([a,b]), \|\cdot\|_1\right)$

Overview of Proof

- The two coarse functions F, G we use that are needed to show a coarse equivalences are the following:
- Define $F : \mathscr{P}([a,b]) \to \mathscr{R}^1([a,b])$ as

$$F(f_P) = Lin(f_P).$$

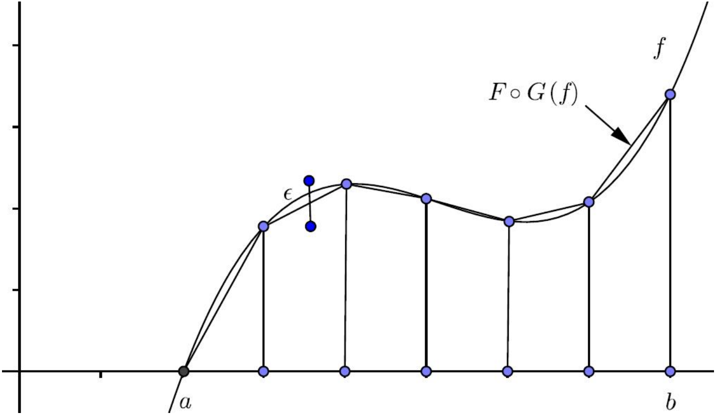
- F is coarse by a previous theorem.
- Let $\varepsilon > 0$. Define $G : \mathscr{R}^1([a,b]) \to \mathscr{P}([a,b])$ as

$$G(f) = f(x), \forall x \in P_N$$

  where $P_n = \left\{a + \frac{b-a}{n} i\right\}_{i=0}^{n}$ and
  $N = \min\{n \in \mathbb{N} \mid U(f, P_n) - L(f, P_n) < \varepsilon\}$.
- G is well defined since $\mathbb{N}$ is well ordered and by Riemann Criterion.

Proof

- We see that $F \circ G(f)$ is close to $f \in \mathscr{R}^1$.



Definition of L^p

- A generalization of $\mathscr{R}([a,b])$.
- We set $L^p(X) = \{f : X \to \mathbb{C} \mid f \text{ measurable} \wedge \|f\|_{L^p} < \infty\}$ where

$$\|f\|_{L^p} = \left(\int_X |f|^p \, dm\right)^{1/p}$$

where m is the lebesgue measure and $X \subset \mathbb{R}$.

Results

Theorem

$L^p([a,b])$ is separable for $1 \leq p < \infty$.

Theorem

$\mathscr{R}^p([a,b])$ is coarse equivalent to $L^p([a,b])$.

Theorem

$\mathscr{P}([a,b])$ is coarse equivalent to $L^1([a,b])$ but not coarse equivalent to $L^p([a,b])$ for $p > 1$.

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Summary

- Basic Intro into Coarse Geometry and some of its properties.
- We showed the existence of non-separable space that is coarse equivalent to a separable space. Hence the topological property of separability is not coarse invariant.

- Outlook
  - Working on finding necessary and sufficient conditions that allows a space X to be coarse equivalent to $L^p(\mathbb{R})$.

Novikov Conjecture

Let M be a closed oriented $n-$dimensional smooth manifold with a map $f : M \to BG$ for some discrete group G and let $\alpha \in H^{n-4\star}(BG;\mathbb{Q})$ be a rational cohomology class. The *higher signature* of M defined by (f,α) is the rational number

$$\sigma_\alpha(M,f) = \langle L_M \cup f^\star \alpha, [M] \rangle \in \mathbb{Q}$$

where $L_M \in H^{4\star}(M;\mathbb{Q})$ is the Hirzebruch L-class of M. Let $h : N \to M$ be a homotopy equivalence of closed oriented smooth manifolds. The **Novikov conjecture** states

$$\sigma_\alpha(N, f \circ h) = \sigma_\alpha(M,f)$$

for all G, f, α and for all homotopy equivalences $h : N \to M$.

Baum-Connes Conjecture

The Baum-Connes conjecture implies the Novikov conjecture. Let Γ be a second countable locally compact group. One can define a morphism

$$\mu_i^\Gamma : RK_i^\Gamma(\underline{E\Gamma}) \to K_i(C_\lambda^\star(\Gamma)),$$

called the **assembly map**, from the equivariant K-homology with Γ-compact support of the classifying space of proper actions $\underline{E\Gamma}$ to the K-theory of the reduced $C^\star$-algebra of Γ. The idex can be 0 or 1. Then the conjecture says that the assemply map μ_i^Γ is an isomorphism.

Contact Info

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P. Nowak.
Coarse Embeddings of Metric Spaces into Banach Spaces
Proceedings of the American Mathematical Society, Vol. 133, No. 9 (Sep., 2005), pp. 2589-2596.

J. Roe.
Lectures on Coarse Geometry
American Mathematical Society, 2003.

P Nowak and G. Yu.
Large Scale Geometry
Lecture notes, 2011.