MATH 5120 - COMPLEX ANALYSIS- LECTURE NOTES

1/20/2015

- $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ where $i = \sqrt{-1}$. This is a field.
  - Addition: $(a + bi) + (c + di) = (a + c) + (b + d)\, i$.
  - Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)\, i$.
- **Remark:** We identify

$$a + bi \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

  we can identify $\mathbb{Q}$ with the set of matrices with matrix addition and multiplication.
- **Definitions/Facts:**
  - $z = a + bi$, and we let $a = \mathrm{Re}z$ (real part) and $b = \mathrm{Im}z$ (imaginary part).
  - $\bar{z} = a - bi$, the conjugate of z.
  - We can write a, b in terms of $z, \bar{z}$ and i:

$$a = \frac{z + \bar{z}}{2} \quad b = \frac{z - \bar{z}}{2i}.$$

  - Here is another easy formula: If $z, w \in \mathbb{C}$ then

$$\begin{aligned} \overline{z + w} &= \bar{z} + \bar{w} \\ \overline{zw} &= \bar{z} \cdot \bar{w}. \end{aligned}$$

  - z is real ($\mathrm{Im}(z) = 0$) iff $z = \bar{z}$.
  - z is imaginary ($\mathrm{Re}z = 0$) iff $z = -\bar{z}$.
  - The modulous (or norm, or absolute value) of z is

$$|z| = \sqrt{a^2 + b^2}.$$

    - It follows that $|z| = |\bar{z}|$ and $z\bar{z} = |z|^2$.
  - The distance between z and w is defined by $|z - w|$. Which is the familiar Euclidean distance.
  - Easy to verify:

$$\begin{aligned} |z \cdot w| &= |z|\,|w| \\ \left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \text{ if } w \neq 0 \\ |\mathrm{Re}z| &\leq |z|\,, \\ |\mathrm{Im}z| &\leq |z|\,. \end{aligned}$$

  - **Triangle Inequality:** $z, w \in \mathbb{C}$ then

$$|z + w| = |z| + |w| + z\bar{w} + \bar{z}w$$

    which implies

$$|z + w| \leq |z| + |w|\,.$$

Proof. We have

$$\begin{aligned} |z+w|^2 &= (z+w)\cdot\overline{(z+w)} \\ &= (z+w)\cdot(\bar{z}+\bar{w}) \\ &= z\bar{z}+w\bar{w}+z\bar{w}+w\bar{z} \\ &\geq |z|^2+|w|^2 \end{aligned}$$

as needed. □

- We can represent z by polar coordinates (r,θ):
$$z = re^{i\theta},\ \text{where } r = |z|\,, \theta = \arg z, \text{argument of } z.$$
If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then
$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$
Note: $e^{i\theta} = \cos\theta + i\sin\theta$.
- $\mathbb{C}$ is not compact. We define it's compactification:
$$\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}\,.\quad \text{Riemann sphere.}$$
Define:
$$\begin{aligned} a+\infty &= \infty+a=\infty, \forall a\in\mathbb{C} \\ b\cdot\infty &= \infty\cdot b=\infty \\ \frac{a}{0} &= \infty, \forall a\in(\mathbb{C}\cup\{\infty\})\setminus\{0\} \\ \frac{b}{\infty} &= 0, \forall b\in\mathbb{C}. \end{aligned}$$
See Figure 1. We use stereogrpahic projection and use
$$\mathbb{S}^2 = \left\{(x_1,x_2,x_3)\mid x_1^2+x_2^2+x_3^2=1\right\}.$$
Where $\varphi : \mathbb{S}^2\setminus\{0,0,1\}\to\mathbb{C}$ where
$$\varphi(x_1,x_2,x_3) = \frac{x_1+ix_2}{1-x_3} = z$$
and
$$|z|^2 = \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{1-x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3}$$
and
$$x_1 = \frac{z+\bar{z}}{|z|^2+1}\qquad x_2 = \frac{z-\bar{z}}{i\left(|z|^2+1\right)}\qquad x_3 = \frac{|z|^2-1}{|z|^2+1}.$$
- Computations:
$$|z|^2+1 = \frac{1+x_3+1-x_3}{1-x_3} = \frac{2}{1-x_3}$$
which implies
$$1-x_3 = \frac{2}{1+|z|^2}$$

and implies

$$x_3 = 1 - \frac{2}{1+|z|^2} = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

- Now φ cna be exended to $S^2 \to \hat{\mathbb{C}}$ by sending $(0,0,1)$ to ∞. So they have the same topology.

Proposition. *The subspace topology of $\mathbb{C}$ from $\hat{\mathbb{C}}$ is the same as the metric topology on $\mathbb{C}$ (from $|z-w|$).*

Proof. It suffices to show that we can define a metric $d(z,w)$ on $\hat{\mathbb{C}}$ and the metric $d(z,w)$ is equivalent to $|z-w|$ for all $z, w \in \mathbb{C}$ in the following sense: Given any real number $c > 0$, there exist $\alpha, \beta > 0$ such that if $|z| \leq c, |w| \leq c$ then

$$\alpha |z-w| \leq d(z,w) \leq \beta |z-w|.$$

For any $z, w \in \hat{\mathbb{C}}$, then $\varphi^{-1}(z) - \varphi^{-1}(w) \in \mathbb{R}^3$ so define

$$d(z,w) = \left|\varphi^{-1}(z) - \varphi^{-1}(w)\right|,$$

where $|\cdot| = |\cdot|_{\mathbb{R}^3}$ and say $\varphi^{-1}(z) = (x_1, x_2, x_3)$ and $\varphi^{-1}(w) = (y_1, y_2, y_3)$. Then

$$\begin{aligned}
d(z,w)^2 &= |(x_1,x_2,x_3) - (y_1,y_2,y_3)|^2 \\
&= |(x_1,x_2,x_3)|^2 + |(y_1,y_2,y_3)|^2 \\
&\quad -2(x_1y_1 + x_2y_2 + x_3y_3), \text{ dot product} \\
&= 2 - 2(x_1y_1 + x_2y_2 + x_3y_3), \text{ norm of 1} \\
&= 2 - 2\frac{\left(1+|z|^2\right)\left(1+|w|^2\right) - 2|z-w|^2}{\left(1+|z|^2\right)\left(1+|w|^2\right)} \\
&= \frac{4|z-w|^2}{\left(1+|z|^2\right)\left(1+|w|^2\right)}.
\end{aligned}$$

Thus

$$d(z,w) \leq 2|z-w|, \forall z, w \in \mathbb{C}.$$

If $|z|, |w| \leq c$, then

$$\frac{2}{1+c^2}|z-w| \leq d(z,w).$$

□

- Now $f : \mathbb{C} \to \mathbb{C}$ defined by $w = f(z)$ where $z, w \in \mathbb{C}$.
- Define
$$\lim_{z \to a} f(z) = A \text{ where } a, A, \in \mathbb{C}.$$
For any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |z-a| < \delta$ implies $|f(z) - A| < \epsilon$.
- Remark:
  - (1) The definition canbe extended to the case a or A is ∞.
  - (2) In the case of $\mathbb{R}$, there are $+\infty$ and $-\infty$, but in $\mathbb{C}$, we only have ∞.
- Note that
$$\lim_{z \to a} f(z) = A \text{ is equivalent to } \lim_{z \to a} \overline{f(z)} = \overline{A}$$
and
$$\text{is also equivalent to } \lim_{z \to a} \mathrm{Re} f(z) = \mathrm{Re} A \text{ and } \lim_{z \to a} \mathrm{Im} f(z) = \mathrm{Im} A.$$

- Also f is **continuous at a** iff
$$\lim_{z\to a} f(z) = f(a).$$
- We say f is **differentiable** at z_0 if
$$f'(z_0 = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \text{ exists.}$$
(Note that $h \in \mathbb{C}$.)
- We say f is **holomorphic** (or **analytic**) at z_0 if there exists a neighborhood V of z_0 such that f is differentiable at every point of V.
- We say f is **holomorphic** (or **analytic**) in a **domain** (connected open set in $\mathbb{C}$) if f is holomorphic at every point of the domain.

Example: $f(z) = (\text{Re} z)^2$ is differentiable at $z = 0$ but not holomorphic at $z = 0$.
Let's compute the derivative
$$\left|\frac{f(h)-f(0)}{h}\right| = \frac{|\text{Re}h|^2}{|h|} \le \frac{|h|^2}{|h|} = |h| \to 0$$
as $|h| \to 0$. This shows $f'(0) = 0$. Let $z_0 \neq 0$. Write $z_0 = x_0 + iy_0$. Let $h \in \mathbb{R}$. Then
$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{(x_0+h)^2 - x_0^2}{h} = 2x_0 + h \to 2x_0$$
as $h \to 0$. Let $h \in \mathbb{R}$ then
$$\frac{f(z_0+ih)-f(z_0)}{ih} = \frac{x_0^2 - x_0^2}{ih} 0 \to 0,$$
so we have different limits. This means $f'(z_0)$ doesn't exist.

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Definition. $f : \mathbb{C} \to \mathbb{C}$ f is differentiable at z_0 if
$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0) \qquad (h \in \mathbb{C})$$
exists. f is holomorphic at z_0 if $\exists \Omega$ s.t. f is differentiable at every point of Ω.

Remark. In fact, any real-valued function of a complex variable has the derivative zero, or the derivative does not exist.

Note that if $h \in \mathbb{R}$ then
$$\frac{f(z_0+h)-f(z_0)}{h} \in \mathbb{R}$$
while
$$\frac{f(z_0+ih)-f(z_0)}{ih} \text{ is purely imaginary.}$$
So it must be zero.

Theorem. *Let $f(z) = u(z) + iv(z)$ be a function defined in a domain $\Omega \subset \mathbb{C}$ where u and v are real-valued. Then f is differentiable at $z_0 \in \Omega$ iff u, v are diffenretiable at z_0, when viewed as function of x and y. i.e. $u(z) = u(x, y)$ and they satisfy the Cauchy-Riemann equations,*
$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Remark. The Cauchy-Riemann equations alone is not sufficient. Say $f(z) = f(x+iy) = \sqrt{|xy|}$???

Proof. ($\Longrightarrow$) Assume f is differentiable at z_0. Then

$$f'(z_0) = A + Bi, \qquad A, B \in \mathbb{R}.$$

Write $z_0 = x_0 + iy_0$and

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - f'(z_0)\right| = \left|\frac{u\left(x_0+h, y_0\right) + iv(x_0+h, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{h} - (A+Bi)\right| \to 0$$

implies that

$$\left|\frac{u\left(x_0+h, y_0\right) - u(x_0, y_0)}{h} - A\right| \to 0 \text{ and } \left|\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} - B\right| \to 0$$

which implies that

$$u_x = A, v_x = B.$$

Now do the same thing but replace h with ih: Write

$$\left|\frac{f(z_0+ih) - f(z_0)}{ih} - f'(z_0)\right| = \left|\frac{u\left(x_0, y_0+h\right) + iv(x_0, y_0+h) - (u(x_0, y_0) + iv(x_0, y_0))}{ih} - (A+Bi)\right| \to 0$$

which implies that

$$v_y = A \text{ and } -u_y = B.$$

Remark:

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

($\Longleftarrow$) Assume u, v differnetiable as functions of x, y and satisfy the Cauchy-Riemann equations. Let $z_0 = x_0 + iy_0$. Denote

$$\begin{aligned} A &= u_x(z_0) = v_y(z_0) \\ B &= u_y(z_0) = -v_x(z_0) \end{aligned}$$

and the same computation is similar except its in the opposite direction. Compute:

$$\lim_{|h+ik|\to 0}\left|\frac{u\left(x_0+h, y_0+k\right) + iv(x_0+h, y_0+k) - (u(x_0, y_0) + iv(x_0, y_0))}{h+ik} - (A+Bi)\right|$$

$$= \lim_{|h+ik|\to 0}\left|\frac{u\left(x_0+h, y_0+k\right) + iv(x_0+h, y_0+k) - (u(x_0, y_0) + iv(x_0, y_0))}{h+ik} - \frac{Ah - Bk + \left(Ak + Bh\right)i}{h+ik}\right|$$

$$\leq \lim_{|h+ik|\to 0}\frac{\left|u\left(x_0+h, y_0+k\right) - u(x_0, y_0) - (Ah - Bk)\right|}{\sqrt{h^2+k^2}} + \lim_{|h+ik|\to 0}\frac{\left|v\left(x_0+h, y_0+k\right) - v(x_0, y_0) - (Ak + Bh)\right|}{\sqrt{h^2+k^2}} \to 0.$$

□

Corollary. *$f(z) = u(z) + iv(z)$ is Holomorphic in a domain iff u, v re differentiable in Ω and satisfy the Cauchy-Riemann equations.*

- We have $f''(z) = (f'(z))'$ similarly $f^{(k)}$ is defined analogously.

Theorem. *(2) If f is holomorphic in a domain Ω , then $f^{(k)}$ exists for all $k \in \mathbb{N}$. Assume the theorem.*

Corollary. *f is holomorphic in a domain Ω iff $u = Ref$ and $v = Imf$ are both C^1 and satisfy the C-R equations.*

Proof. If f is holomorphic, then write

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

By Theorem 2, $f''(z)$ exists so tht

$$\begin{aligned} f''(z) &= u_{xx} + iv_{xx} = v_{xy} - iu_{xy} \\ &= u_{yx} - iv_{yx} = -u_{yy} - iv_{yy}. \end{aligned}$$

Which implies $u, v \in C^1$ as needed. □

Corollary. *$f = u + iv$ holomorphic in Ω. Then u, v are harmonic i.e.*

$$\Delta u := u_{xx} + u_{yy} = 0$$

and

$$\Delta y := v_{xx} + v_{yy} = 0.$$

Proof. By Cauchy Riemann, we have that $u_x = v_y$ and $u_y = -v_x$. Which implies that

$$u_{xx} = v_{yy} = v_{xy} = -u_{yy}$$

because $u, v \in C^2$ by theorem 2. □

Definition. If u, v are harmonic and satisfy the C-R equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x, \end{aligned}$$

then we call v is the conjugate harmonic of u.
($-u$ is the conjugate harmonic of v)

Corollary. *Suppose Ω is simply-connected. ($\pi_1(\Omega) = 0$ e.g. a unit disk). Let u be harmonic in Ω. Then there exists a conjuagte harmonic v in Ω. Consequently, $f = u + iv$ is holomorphic in Ω.*

Proof. Recall in Calculus: $\mathbb{F} = (P, Q)$ is conservative vector field if $\mathbb{F} = \Delta f$. A theorem says if $P_y = Q_x$ in a simply connected domain,then $\mathbb{F} = \nabla f$. Define $\mathbb{F} = (-u_y, u_x)$ then

$$(-u_y)_y = (u_x)_x.$$

But since Ω is simply connected $\implies \mathbb{F} = \nabla V = (v_x, v_y)$. Which implies u, v satisfy the Cauchy Riemann equations. □

- Now $z = x + iy$ and $\bar{z} = x - iy$ so that

$$x = \frac{z + \bar{z}}{2} \text{ and } y = \frac{z - \bar{z}}{2i}, -i\frac{z - \bar{z}}{2}.$$

- We compute

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

  and

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

  with

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(u_x + iv_x + i\left(u_y + iv_y\right)\right) \\ &= \frac{1}{2}\left(\left(u_y - v_y\right) + i\left(v_x + u_y\right)\right), \end{aligned}$$

So that

$$\text{C-R equations} \iff \frac{\partial f}{\partial \bar{z}} = 0$$

f is holomorphic implies f is independent of $\bar{z}$.

Definition. $f \in C^k(\Omega)$ if $u, v \in C^k(\Omega)$ where $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$.

Corollary. *f is holomorphic in Ω iff $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \bar{z}} = 0$ in Ω.*

Example. Take

$$f(z) = (\mathrm{Re} z)^2 = \frac{(z+\bar{z})^2}{4}$$

we have that

$$\frac{\partial f}{\partial \bar{z}} = \frac{(z+\bar{z})}{2}$$

We have f is not holomorphic.

Example. $f(z) = \log|z|$ is not Holomorphic: Compute

$$\begin{aligned} f(z) &= \log|z| \\ &= \frac{1}{2}\log|z|^2 \\ &= \frac{1}{2}\log(z\bar{z}). \end{aligned}$$

In $\mathbb{C}\setminus\{0\}$ we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{z}{2z\bar{z}} = \frac{1}{2\bar{z}}.$$

Definition. f is harmonic if

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \quad \left(\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}(f_{xx} + i f_{yy}) \text{ no } i? \right).$$

Remark. A holomorphic function is harmonic.

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$$

in $\mathbb{C}\setminus\{0\}$.

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Definition. A complex-valued function f is holomorphic in a domain Ω if $f \in C^1(\Omega)$ and

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

- Sequence/Series of complex numbers
- Sequence/Series of complex valued function.
- power series: $\sum_{n=1}^{\infty} a_n z^n$
- Polylonmial: $P(z) = a_n z^n + \cdots + a_1 z$.
- Rational Functions:
  - $R(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials. is holomoprihc on the domain $\{z \in \mathbb{C} \mid Q(z) \neq 0\}$.

- $\{a_n\}$ sequence of complex numbers is said to have a limt $A \in \mathbb{C}$ if
$$\lim_{n\to\infty} a_n = A.$$
  - Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then
$$|a_n - A| < \epsilon.$$
- Cauchy criterion
  - $\{a_n\}$ coverges if and only if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$ then
$$|a_n - a_m| < \epsilon.$$

Proof. Assume $\lim_{n\to\infty} = A$ then
$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < 2\epsilon$$
for large enough n and m.

Now assume $a_n = \beta_n + \gamma_n i$ where $\beta_n, \gamma_n \in \mathbb{R}$. Then
$$\begin{aligned} |a_n - a_m| &= |\beta_n - \beta_m + (\gamma_n - \gamma_m)\, i| \\ &= \sqrt{(\beta_n - \beta_m)^2 + (\gamma_n - \gamma_m)^2} < \epsilon \end{aligned}$$
which implies β_n and γ_n are Cauchy sequewnces (of real nuimbers).

This implies that ,$\beta_n \to B$ and $\gamma_n \to C$, so that $a_n \to B + iC$. □

- Infinite Series: $\sum_{n=1}^{\infty} a_n$ where partial sum is $S_n = \sum_{k=1}^{n} a_n$.

Definition. $\sum a_n$ is said to be convergent if the sequence $\{S_n\}$ converges.

Proposition. *A series $\sum a_n$ converges iff given $\epsilon > 0$ there esists $N \in \mathbb{N}$ such that if $n \geq N$ and $p \geq 0$ then*
$$|a_n + a_{n+1} + \cdots + a_{n+p}| < \epsilon.$$

Corollary. *If a series $\sum a_n$ convefrges, then* $\lim_{n\to\infty} a_n = 0$.

Definition. $\sum a_n$ is absolutely covergent if $\sum |a_n|$ converges.

Corollary. *If a series $\sum a_n$ converges absolutely, thenit converges.*

Proof. Triangle inequality:
$$|a_n + \cdots a_{n+p}| \leq |a_n| + \cdots + |a_{n+p}|\,.$$
□

- f_n a sequence of complex-valued functions defined on $E \subset \mathbb{C}$.
- f_n converges (pointwise) to a function f on E, if $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in E$.
- f_n converges uniformly to f. if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then
$$\|f_n - f\|_E = \sup_{z\in E} |f_n(z) - f(z)| < \epsilon.$$

Proposition. *If f_n is a sequene of continus functions on $E \subset \mathbb{C}$ and d_n convefrges uniformly to $f$on E, then f is continuous.*

Proof. Take $z, w \in \mathbb{C}$ then
$$\begin{aligned} |f(z) - f(w)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(w)| + |f_n(w) - f(w)| \\ &\leq 3\epsilon. \end{aligned}$$
□

- Cauchy Criterion for sequence of functions.

Proposition. *f_n uniformly converges on E iff given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|f_n - f_m\|_E < \epsilon.$$

Proof. Assume f_n uniformly converges. Then use triangle inequality like before.

Assume given $\epsilon < 0, \exists N$ s.t $n, m \geq N$ implies $\|f_n - f_m\|_E < \epsilon$. Fice $\omega \in E$, we have $f_n(\omega) \to f(\omega)$. Given $\epsilon > 0, \exists N_1 = N_1(\omega) \geq N$ such that

$$|f_{N_1}(\omega) - f(\omega)| < \epsilon.$$

Fo any $n \geq N$ we have

$$\begin{aligned} |f_n(\omega) - f(\omega)| &\leq |f_n(\omega) - f_{N_1}(\omega)| + |f_{N_1}(\omega) - f(\omega)| \\ &\leq 2\epsilon. \end{aligned}$$

which implies

$$\|f_n - f\|_E \leq 2\epsilon.$$

□

Corollary. *$\{f_n\}$ a sequence of functions. $\{a_n\}$ convergent sequenve of numbers. If $\|f_n - f_m\|_E \leq |a_n - a_m|$ then f_n uniformly converges on E.*

- $\sum f_n$ is a infinite series of complex-valued functions.
- $\sum f_n$ converges if the sequence of partial sums $S_n = \sum_{k=1}^{n} f_k$ converges.
- $\sum f_n$ converges uniformly if the sequence of partial sums S_n converges uniformly on E.

Corollary. *(Weirstraus M-test)$\sum f_n(z)$ on E and $\sum a_n$ convergent series of real numbers. If there exists $M > 0$ such that $\|f_n\|_E \leq M|a_n|$ for n sufficienly large, then $\sum_n f_n$ converges uniformly.*

Proof. By Cauchy criterion

$$\begin{aligned} \|S_{n+p} - S_{n+l}\|_E &= \|f_n + \cdots + f_{n+p}\|_E \\ &\leq \|f_n\|_E + \cdots + \|f_{n+p}\|_E \\ &\leq M(a_n + \cdots + a_{n+p}) \\ &\leq M\epsilon. \end{aligned}$$

□

Power Series:

$$\sum_{n=0}^{\infty} a_n z^n \quad \left(z^0 := 1 \text{ by converntion}\right).$$

where $a_n \in \mathbb{C}$.

- **Example:** (geometric series)
  - $\sum_{n=0}^{\infty} z^n$ and $\sum_{j=0}^{n} z^j = \frac{1-z^{n+1}}{1-z}$.
  - Diverges when $|z| \geq 1$ because $\lim_{n\to\infty} \left|z^j\right| \neq 0$.
  - Converges when $|z| < 1$, in fact

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

- See that

$$\left|\sum_{j=0}^{n} z^j - \frac{1}{1-z}\right| = \frac{|z^{n+1}|}{|1-z|} \to 0.$$

Theorem. *(Abel) For every power series $\sum_n a_n z^n$, there exists a number R with $0 \leq R \leq \infty$, given by Hadamard's formula:*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

satisfying.

(1) The series converges absolute for all $|x| < R$. For any $0 < \rho < R$, the series converges uniformly on $\{|z| \leq \rho\}$.

(2) For any $|z| > R$, the series diverges. (In fact, $\{a_n z^n\}$ is unbounded).

(3) On $\{|z| < R\}$ the series is holomorphic. The derivative is obtained by term-wise derivative and the derivaed series has the same radus of convegrence.

- Limsup?
- $\{s_n\}$ a sequence of real numbers.
- $M_k = \sup\{s_k, s_{k+1}, \dots\} = \sup_{n\geq k} s_n$. M_k is a decreaing sequence and bounde below. This means that $\lim_{k\to\infty} M_k$ exists.
- Hence defined $\limsup_{n\to\infty} s_n := \lim_{k\to\infty} M_k$.

2/3/2015

- Today
  - 1. Holomorphic functions
  - 2. Series.
  - 3. Exp and Trig functions

Theorem. *Let $\Omega \subset \mathbb{C}$ be a domain. f is holomorphic in Ω if and only if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \bar{z}} = 0$ in Ω.*

- Let $f = u + iv$ where u, v are real-valued functions. Then f is holomorphic in $\Omega \implies \Delta u = 0$ and $\Delta v = 0$ in Ω.
- **Example:** Show that

$$u(x,y) = \log \frac{\sqrt{x^2 + (y+1)^2}}{\sqrt{x^2 + (y-1)^2}}$$

is harmonic in the domain $\Omega = \left\{ (x,y) \in \mathbb{R}^2 \mid y > 0, x^2 + (y-1)^2 > 1 \right\}$.

Proof. Let $z = x + iy$. Then

$$|z \pm i|^2 = x^2 + (y \pm 1)^2.$$

So

$$\begin{aligned} u &= \frac{1}{2} \log \frac{|z+i|^2}{|z-i|^2} \\ &= \frac{1}{2} \log |z+i|^2 - \frac{1}{2} \log |z-i|^2. \end{aligned}$$

Note that

$$\Delta u = u_{xx} + u_{yy} = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

Since $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Then

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \bar{z}} \left(\log |z+i|^2 \right) &= \frac{\partial}{\partial z} \left(\frac{1}{|z+i|^2} \cdot (z+i) \right) \text{ Because } \frac{\partial}{\partial \bar{z}} \left(|z|^2 \right) = \frac{\partial}{\partial \bar{z}} (z\bar{z}) = z. \\ &= \frac{\partial}{\partial z} \left(\frac{1}{\bar{z} + \bar{i}} \right) \\ &= 0 \end{aligned}$$

Then

$$|z+i|^2 = (z+i)(\bar{z} + \bar{i}).$$

Hence $\Delta u = 0$. □

- Note that $\frac{\partial^2}{\partial z \partial \bar{z}} \log |z|^2 = 0$ for any $z \neq 0$.

Theorem. *(Abel) Given series $\sum_{n=0}^{\infty} a_n z^n$ there exists a number*

$$R = \frac{1}{\limsup_{k \to \infty} \sqrt[k]{|a_k|}} \in [0, +\infty]$$

called the radius of convergence satisfies the following

(1) The series converges absolutely absolutely for all $|z| < R$. For all $\rho \in [0, R]$ the series converges uniformly on $\{|z| \leq \rho\}$.

(2) For all $|z| > R$, then the term $a_n z^n$ are unbounded and the series is divergent.

(3) In $\{|z| < R\}$, the series is a holomorphic function, denoted by $f(z)$. Then $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, where $\sum n a_n z^{n-1}$ has the same radius of convergence.

- **Recall:** Let $\{s_n\}$ be bounded real sequence and $\beta = \limsup_{n\to+\infty} s_n$. Then
  - i) $\forall \epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $s_n < \beta + \epsilon$ for all $n \geq N_1$.
  - ii) For all $\epsilon > 0$ and for all $N_2 \in \mathbb{N}$ there exists $m \geq N_2$ such that $s_m < \beta - \epsilon$.

Proof. Part (1) Assume wlog $R > 0$. Fiz z with $|z| < R$. Pick $\rho > 0$ such that $|z| < \rho < R$. ($\frac{1}{R} < \frac{1}{\rho}$) Then we want to bound $\sum_n |a_n| |z|^n$. Since $\frac{1}{R} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} < \frac{1}{\rho}$.Then by property i) of the limsup we have

$$|a_n|^{\frac{1}{n}} < \frac{1}{\rho} \;\forall n \geq N_1.$$

To show uniform convergence, we pick $\rho_1 \in (\rho, R)$. Then by the previous argument

$$\begin{aligned}\sum |a_n| |z|^n &\leq \sum \frac{|z|^n}{\rho_1^n} \\ &\leq \sum \frac{\rho^n}{\rho_1^n} \\ &\leq \frac{1}{1 - \frac{\rho}{\rho_1}} < +\infty,\end{aligned}$$

as needed.

Part (2) Leave as an exercise.

Part (3)

First to show $\sum n a_n z^{n-1}$ has radius radius of convergence R. Note $z \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n a_n z^n$, then we claim that this implies that

$$\sum n a_n z^{n-1} \text{ and } \sum n a_n z^n.$$

have the same radius of convergence. This is easy to see. Now we compute

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{n\,|a_n|} &= \limsup_{n\to\infty} \sqrt[n]{n}\,\sqrt[n]{|a_n|} \\ &= \lim_{n\to\infty} \sqrt[n]{n} \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R}.\end{aligned}$$

This follows since $n^{\frac{1}{n}} = e^{\frac{1}{n}\log n} \to e^0 = 1$. Also used Fact: $s_n t_n$ real and $\lim t_n = T$. Then $\limsup s_n t_n = T \limsup s_n$.

Let

$$g(x) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Want to show that $f'(z) = g(z)$ for all $|z| < R$. We just need to show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = g(z)$$

and this would have shown that this series is holomorphic in the radius of convergence. Write

$$f(z)=\sum_{k=0}^{n}a_nz^n+\sum_{k=n+1}^{\infty}a_nz^n=f_n(z)+R_n(z),$$

and

$$g(z)=\cdots=g_n(z)+\tilde{R_n}(z).$$

and where

$$g_n(z)=\sum_{k=1}^{n}na_nz^{n-1}.$$

Now

$$\begin{aligned}\left|\frac{f(z+h)-f(z)}{h}-g(z)\right| &\leq \left|\frac{f_n(z+h)-f(z)}{h}-g_n(z)\right| \\ &\quad+\left|\frac{R_n(z+h)-R_n(z)}{h}\right|+\left|g_n(x)-g(z)\right|.\end{aligned}$$

The first and last terms are clearly less than some ϵ. So we must estimate the middle term $\left|\frac{R_n(z+h)-R_n(z)}{h}\right|$. Using the formula

$$A^k-B^k=(A-B)\left(A^{k-1}+\cdots+B^{k-1}\right),$$

we have that if $|h|<\delta$, δ small enough so that $|z+h|<|z|+\delta$ then we can estimate

$$\begin{aligned}\left|\frac{R_n(z+h)-R_n(z)}{h}\right| &= \frac{1}{|h|}\left|\sum_{k=n+1}^{\infty}a_k(z+h)^k-\sum_{k=n+1}^{\infty}a_kz^k\right| \\ &\leq \frac{1}{|h|}\sum_{k=n+1}^{\infty}|a_k|\left|(z+h)^k-z^k\right| \\ &= \frac{1}{|h|}\sum_{k=n+1}^{\infty}|a_k|\,|h|\left|(z+h)^{k-1}+\cdots+z^{k-1}\right| \\ &= \sum_{k=n+1}^{\infty}|a_k|\left(|z|+\delta\right)^{k-1} \\ &\leq \sum_{k=n+1}^{\infty}\frac{k\left(|z|+\delta\right)^{k-1}}{R^k} \\ &< \epsilon\end{aligned}$$

where this is finite since this is a convergent geometric series. This finishes the proof. □

3. Exponential functions:
Define

$$e^z=\sum_{k=0}^{\infty}\frac{z^k}{k!}$$

for all $z \in \mathbb{C}$. Note that

$$\begin{aligned} \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k!}} &= \limsup_{k\to\infty} \exp\left(-\frac{1}{k}\sum_{j=1}^{k}\log j\right) \\ &= 0 \end{aligned}$$

because

$$\lim_{k\to+\infty}\left(-\frac{1}{k}\sum_{j=1}^{k}\log j\right) = \lim_{k\to\infty}(-\log k) = \infty,$$

where we used the Fact: if $s_n \to L$ then $\lim \frac{s_1+\cdots+s_n}{n} = L$. By Abel's Theorem we have that

$$(e^z)' = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z.$$

Also e^z is holomopthic on $\mathbb{C}$.

Proposition. *(Addition formula)*

$$e^{a+b} = e^a e^b, \forall a, b \in \mathbb{C}.$$

Proof. Let $g(z) = e^z e^{c-z}$ where $c \in \mathbb{C}$. I claim: $g(z) \equiv g(0) = e^c$. Then the result follows from letting $z = a$ and $c = a + b$. It remains to prove the claim. To show a function is cosntant we differentiate. We have

$$\begin{aligned} g'(z) &= e^z e^{c-z} \\ &= (e^z)' e^{c-z} + e^z \left(e^{c-z}\right)' \\ &= e^z e^{c-z} - e^z e^{c-z} \\ &= 0. \end{aligned}$$

By partials with $\bar{z} = 0$ hence $g \equiv$consntat. □

Corollary. *For all $z \in \mathbb{C}$.*
(1) $e^z e^{-z} = 1$. In particular, $e^z \neq 0$.
(2) $\overline{e^z} = e^{\bar{z}}$. Need to prove this.
This implies: ($|e^z| = e^{Re z}$ and $\left|e^{x+iy}\right| = e^x$ and $\left|e^{iy}\right| = 1$).

2/5/2015

- Last time we had

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}, z \in \mathbb{C}$$

  and $R = +\infty$.
- We had $(e^z)' = e^z$.
- We had the following proposition:

Proposition. *(Addition formula)*

$$e^{a+b} = e^a e^b, \forall a, b \in \mathbb{C}.$$

Proof. Recall the idea was to let $g(z) = e^z e^{c-z}, \forall z$ where $c \in \mathbb{C}$. Note $g'(z) = 0$. By Homeowrk 1.4(a) we get $g(z) \equiv g(0) = e^c$. Putting $c = a + b$ and $z = a$ yields the result. □

Corollary. *For all $z \in \mathbb{C}$.*
(1) $e^z e^{-z} = 1$. In particular, $e^z \neq 0$.
(2) $\overline{e^z} = e^{\bar{z}}$. Consequently we get ($|e^z| = e^{Re z}$ and $\left|e^{x+iy}\right| = e^x$ and $\left|e^{iy}\right| = 1$).

Proof. Only need to show that $\overline{e^z} = e^{\bar{z}}$. We have that

$$\left|\overline{e^z} - e^{\bar{z}}\right| \leq \left|\overline{e^z} - \overline{\sum_{n=0}^{N} \frac{z^n}{n!}}\right| + \left|\overline{\sum_{n=0}^{N} \frac{z^n}{n!}} - e^{\bar{z}}\right|,$$

and we want to use $\overline{a+b} = \bar{a} + \bar{b}$ for $a, b \in \mathbb{C}$. Now first we see that the first term becomes

$$\begin{aligned}\left|\overline{e^z} - \overline{\sum_{n=0}^{N} \frac{z^n}{n!}}\right| &= \left|\overline{e^z - \sum_{n=0}^{N} \frac{z^n}{n!}}\right| \\ &= \left|e^z - \sum_{n=0}^{N} \frac{z^n}{n!}\right| \\ &\to 0 \text{ as } N \to \infty.\end{aligned}$$

Now since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ then $e^{\bar{z}} = \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!}$, so that the second term

$$\left|\overline{\sum_{n=0}^{N} \frac{z^n}{n!}} - e^{\bar{z}}\right| = \left|\sum_{n=0}^{N} \frac{\bar{z}^n}{n!} - e^{\bar{z}}\right| \to 0 \text{ as } N \to \infty.$$

Thus

$$\left|\overline{e^z} - e^{\bar{z}}\right| = 0$$

as needed.

Now note

$$|e^z|^2 = e^z\overline{e^z} = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2\text{Re}(z)}$$

so that

$$|e^z| = e^{\text{Re}(z)}$$

as needed. □

Trig Functions:

We defined the following:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \quad (\star)$$

Then by the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ we have the following representations

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \qquad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

By $(\star)$ we have that

$$\begin{aligned}\sin^2 z + \cos^2 z &= 1, \ z \in \mathbb{C} \\ e^{iz} &= \cos z + i \sin z \ z \in \mathbb{C}.\end{aligned}$$

Let $y \in \mathbb{R}$ then we have **Euler's formula**

$$e^{iy} = \cos y + i \sin y. \quad \text{(Euler)}.$$

Then if $z = x + iy$ we have

$$e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y).$$

Proposition. *1. If $x \in \mathbb{C}$ satisfies $e^z = 1$ then $z = 2k\pi i$ for some $k \in \mathbb{Z}$.*

Proof. For $x, y \in \mathbb{R}$, write $z = x + iy$ and $1 = e^z = e^x e^{iy}$ then take the norm

$$1 = \left|e^x e^{iy}\right| = |e^x| = e^x$$

which implies that $x = 0$. But then

$$e^{iy} = \cos y + i \sin y = 1$$

hence we have the following two equations $\cos y = 1$ and $\sin y = 0$. Solving these yields $y \in 2\pi\mathbb{Z}$. □

Proposition. *2. Let $z = x + iy$. Then*

$$\begin{aligned} |\cos z|^2 &= \cosh^2 y - \sin^2 x, \\ |\sin z|^2 &= \sinh^2 y + \sin^2 x. \end{aligned}$$

In particular, $|\sin z|$ and $|\cos z|$ are unbounded.

- Recall that the hyperbolic functions are:

$$\begin{aligned} \cosh y &= \frac{e^y + e^{-y}}{2}, \\ \sinh y &= \frac{e^y - e^{-y}}{2} \end{aligned}$$

Proof. Write $|\cos z|^2 = \cos z \overline{\cos z}$ and

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iy}e^{ix} + e^y e^{-ix}}{2}$$

and

$$\overline{\cos z} = \frac{e^{-y}e^{-ix} + e^y e^{ix}}{2},$$

then we leave the rest of the calculation as an exercise. □

3. Logarithms

Try to define $\log z$ on $\mathbb{C}$ and we'll see that this is not well defined. Let $w = \log z$ to be a root of $e^w = z$. Write $z = |z| e^{i\theta}$. Let $w = x + iy$ for $x, y \in \mathbb{R}$. For $e^w = z$ we have

$$\begin{aligned} e^w = z \quad &\Longrightarrow \quad e^x e^{iy} = |z| e^{i\theta} \\ &\Longrightarrow \quad e^x = |z| \\ &\Longrightarrow \quad x = \log |z|. \end{aligned}$$

Then

$$e^{iy} = e^{i\theta}, \ e^{i(y-\theta)} = 1.$$

By Proposition 1, $y = \theta + 2k\pi, \forall k \in \mathbb{Z}$. So

$$\log z = \log |z| + i (\theta + 2k\pi), k \in \mathbb{Z}$$

is a "multi-valued function", which is not a function. If we define $\log z_1 = \log |z_1| + i\theta_1$ then moving along Γ (a curve about the origin and z_1 in the this curve and $\theta \mapsto \theta + 2\pi$) we get

$$\log z_1 = \log |z_1| + i2\pi$$

which is not well defined.

But if we restrict $\log z$ to a domain G not containing the origin O. (Draw a domain G not containing the origin) define for all $z \in G$

$$\log z := \log|z| + i\theta \text{ where } -\pi < \theta < \pi$$

then this function is well-defined. (So a loop in this domain keeps the angle unchanged because it does not cross the origin.) We call this a branch, and the origin is called a branch point. Also we could restrict $\theta \in (\pi, 2\pi), \dots$.

Definition. A point z_0 is called a branch point of a function $f(z)$, if $f(z)$ changes its value as one starts out at a point, traces a closed path enclosing z_0 and returns to the starting point.

Example. For $f(z) = \log z$, $z_0 = 0$ is a branch point, any $z \neq 0, z \in \mathbb{C}$, is not a branch point.

Definition. Let G be a domain in $\mathbb{C}$. If $f : G \to \mathbb{C}$ is a continuous function such that $e^{f(z)} = z, \forall z \in G$, then f is called a <u>branch</u> of $\log z$ on G.

- Let f be a branch of $\log z$ on G. Let $g = f + 2k\pi i$ for some $k \in \mathbb{Z}$. Then
$$e^g = e^f e^{2k\pi i} = e^f = z.$$

Proposition. *3. Let f be a branch of $\log z$ on G. Then the set of all branches of $\log z$ on G is precisely $\{f(z) + 2k\pi i; k \in \mathbb{Z}\}$.*

Proof. Let g be branch of $\log z$ on G. For each $z \in G$, $e^{g(z)} = e^{f(z)} = z$. By Proposition 1, we have $g(z) = f(z) + 2\pi k_z i$ for $k_z \in \mathbb{Z}$. Want to get a k independent of $z \in G$. Let

$$h(z) = \frac{g(z) - f(z)}{2\pi i}.$$

Then $h \in C^0(G)$, $h(G)$ connected. Note $h(G) \subset \mathbb{Z}$ discrete, so the image must be one point, otherwise it wouldn't be connected. Hence, $h(G) = \{k_0\}$ for some $k_0 \in \mathbb{Z}$. i.e. $g(z) = f(z) + 2k_0\pi i$. □

- Question: What is the largest possible domain for a well defined $\log z$?
- Answer: $\mathbb{C} \setminus \{z = x + yi \mid z \leq 0, y = 0\}$.
- Another way to answer the question:
  - $\log z$ has a branch point $z = 0$ in $\mathbb{C}$.
  - $\log z$ also has a branch point at $z = \infty$ in $\hat{\mathbb{C}}$. To see this, let $z = \frac{1}{w}$, then $w = 9 \mapsto z = \infty$. so that $\log z = -\log w$ which implies that $w = 0$, $(z = \infty)$ is a branch point.
  - Let $G = \hat{\mathbb{C}} \setminus \{\text{a curve joining } 0 \ \& \ \infty\}$. Then there exists a branch of $\log z$ on G.
  - From now on, we call
  $$\log|z| + i\theta, \quad -\pi < \theta < \pi$$
  on $G_1 = \mathbb{C} \setminus \{z = x + yi \mid z \leq 0, y = 0\}$ the <u>**principal branch**</u> of $\log z$. Sometimes we denote $\log z = \log|z| + i\theta$.

Remark. Given G and $f(z)$ branch of $\log z$ such that $e^{f(z)} = z$ and $(f \in C^0(G))$. We can show that

$$f'(z) = \frac{1}{z} \text{ on } G, \quad (\star\star)$$

hence, f is analytic. That is, $(\log z)' = \frac{1}{z}$ on G_1.

- $(\star\star)$ follows from the following.

Lemma. *(IVT) Let U,V be open sets in $\mathbb{C}$. Suppose that $f:U\to\mathbb{C}$ and $g:V\to\mathbb{C}$ are both continuous, $f(U)\subset g(V)$ and*

$$g\left(f\left(z\right)\right)=z.$$

If g is differntiable and $g'(z)\neq0$ for all $z\in V$, then f is differentiable and

$$f'(z)=\frac{1}{g'\left(f\left(z\right)\right)}.$$

Apply to $g=e^{z}$.

- Define for all $a,b\in\mathbb{C}$ then following

$$a^{b}=e^{b\log a}.$$

  - i) if $a>0,a\in\mathbb{R}$ then a^{b} is signle-valued.
  - ii) If $b\in\mathbb{Z}$, then a^{b} signle-valued. ($e^{2i\pi kb}=1$) if $b\in\mathbb{Q}$?

2/10/2015

- A Quick way to solve problem 1 from homework 1.
  - Let $F=g+ih$. Then

$$\begin{aligned}f &= \log\left(h^{2}+g^{2}\right)\\ &= \log\left|F\right|^{2}.\end{aligned}$$

Since $h\neq0$ in disk then $F\neq0$. Then define the operator

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{z}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\\ \frac{\partial}{\partial\bar{z}} &= \frac{1}{z}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).\end{aligned}$$

Then

$$4\frac{\partial^{2}}{\partial z\partial\bar{z}}=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y}.$$

It suffices to show that $\frac{\partial^2}{\partial z\partial\bar{z}}\log|F|^2=0$. But then

$$\begin{aligned}\frac{\partial}{\partial\bar{z}}\log|F|^2 &= \frac{1}{|F|^2}\frac{\partial}{\partial\bar{z}}|F|^2\\ &= \frac{1}{|F|^2}\frac{\partial}{\partial\bar{z}}F\bar{F}\\ &= \frac{1}{|F|^2}\left(\bar{F}\frac{\partial}{\partial\bar{z}}F+F\frac{\partial}{\partial\bar{z}}\bar{F}\right)\\ &= \frac{1}{|F|^2}\left(\bar{F}\frac{\partial F}{\partial\bar{z}}+F\frac{\partial\bar{F}}{\partial\bar{z}}\right)\\ &= \frac{1}{|F|^2}F\cdot\frac{\partial\bar{F}}{\partial\bar{z}},\ \ (\frac{\partial F}{\partial\bar{z}}=0)\\ &= \frac{1}{F\bar{F}}F\cdot\frac{\partial\bar{F}}{\partial\bar{z}}\\ &= \frac{1}{\bar{F}}\frac{\partial\bar{F}}{\partial\bar{z}}.\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial^2}{\partial z\partial\bar{z}}\log|F|^2 &= \frac{\partial}{\partial z}\left(\frac{1}{\bar{F}}\frac{\partial\bar{F}}{\partial\bar{z}}\right)\\ &= \frac{\bar{F}\frac{\partial^2\bar{F}}{\partial z\partial\bar{z}}-\frac{\partial\bar{F}}{\partial\bar{z}}\frac{\partial\bar{F}}{\partial z}}{\bar{F}^2}\\ &= \frac{\bar{F}\cdot 0-\frac{\partial\bar{F}}{\partial\bar{z}}\cdot 0}{\bar{F}^2}\\ &= 0\end{aligned}$$

since $\frac{\partial\bar{F}}{\partial z}=\overline{\frac{\partial F}{\partial\bar{z}}}=\bar{0}=0$ and since $\bar{F}$ is holomorphic then it is complex harmonic so that $\frac{\partial^2\bar{F}}{\partial z\partial\bar{z}}=0$ as needed.

- Delicate Way:
  - In general $\log|F|^2\neq\log F+\log\bar{F}$. where $\log z$ is a multivalued function on $\mathbb{C}$.
  - Note that $F=g+ih$ and $h\neq 0$. Then

$$F\left(\{|z|<1\}\right)\subset\mathbb{C}\backslash\left\{\mathrm{Re}z\leq 0,\mathrm{Im}z=0\right\}=G_1.$$

    On g_1 take $\log z$ the principal branch of G_1. By chain rule $\log F$ is holomotphic on $\mathbb{D}$.
  - Then $\log|F|^2=\log F+\log\bar{F}=2\mathrm{Re}(\log F)$ is harmonic.

Last time:

- Define $a^b=e^{b\log a}$. So in general this is a multivalued function. But there are cases where we have single valued function.
  - 1) if $a>0$ $a\in\mathbb{R}$ then $\log a$ is single-valued , so is a^b.
  - 2) If $b\in\mathbb{Z}$, then a^b is single valued. This is because the log is differered by $2k\pi ib$, since $e^{2k\pi ib}=1$.
  - 3) if $b\in\mathbb{Q}$, and $b=\frac{p}{q}$ reduced form. $(p,q\in\mathbb{Z})$ then $a^b=e^{\frac{p}{q}\log a}$ and $\log a=\log|a|+\arg a+2k\pi i$. So a^b is $q-$valued function.

Example: Suppose $b = \frac{1}{2}$. Consider $z^{\frac{1}{2}} = e^{\frac{1}{2}\log z}$ is a two-valued function. Take the principal branch of $z^{\frac{1}{2}}$ to be $\sqrt{z} = |z|^{\frac{1}{2}} e^{i\theta/2}$ for $-\pi < \theta < \pi$ on G_1. (Another branch on G_1 is $-|z|^{\frac{1}{2}} e^{-i\theta/2} = -\sqrt{z}$)

Remark: The branch points of $\sqrt{z}$ are $\{0, \infty\}$. Pick any curve γ joint 0 and ∞. Then $\sqrt{z}$ has a branch on $\hat{\mathbb{C}} \backslash \gamma$. .

Remark$\star$: Riemann's idea: Idea of 5121- Complex 2.

Complex Integration:

- Our goal is to show:

Theorem. *0. (Cauchy Integral formula). Let Ω be a bounded domain in $\mathbb{C}$ with C^1 boundary $\partial\Omega$. Then for all holomprihc f on Ω with $f \in C^1\left(\bar{\Omega}\right), z$*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi, \quad \forall z \in \Omega.$$

- A <u>curve</u> $\gamma : [a, b] \to \mathbb{C}$ is a continuous function. (i.e. if $\gamma(t) = u(t) + iv(t)$ then $u, v \in C^0([a, b])$).
- A curve γ is simple if $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$.
- A curve γ is closed if $\gamma(a) = \gamma(b)$.
- The direction of $\gamma(t)$ is the direction along which t is increasing.
- A curve is Jordan if it is simple and closed.
- A curve γ is of $C^1([a, b])$, if $\gamma'(t) = u'(t) + iv'(t)$ exists and continuous on $[a, b]$.
- A curve δ is of piecewise C^1 if $\gamma \in C^0([a, b])$, and γ' exists except for $t_0, t_1, \ldots, t_m \in [a, b]$. At each t_j, $\gamma'_+(t_j) = \gamma'(t_j+)$, and $\gamma'_-(t_j) = \gamma'(t_j-)$.
- Let $f(t) = u(t) + iv(t)$ be a continuous function on $[a, b]$. Define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

- This implies

$$\overline{\int_a^b f(t)dt} = \int_a^b \overline{f(t)}dt.$$

- This means

$$c\int_a^b f(t)dt = \int_a^b cf(t)dt, \forall c \in \mathbb{C}$$

  and

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt.$$

- Let γ be a C^1 curve on $[a, b]$, and $f(\gamma(t))$ be a continuous function on $[a, b]$. Define

$$\int_\gamma f(z)dz = \int_a^b f(z(t))\, z'(t)dt$$

  where $z = z(t) = \gamma(t)$.

Example2: Let γ be the boundary of the ball $B_R(0)$ in $\mathbb{C}$. Find $\int_\gamma f(z)dz$ for nay continuous function $f(z)$. For all $z \in \partial B_R(0)$ then

$$z = Re^{i\theta} \quad 0 \leq \theta < 2\pi.$$

So γ has the representation $v(\theta) = z(\theta) = Re^{i\theta}$ and $z'(\theta) = Re^{i\theta}i$. Hence

$$\int_\gamma f(z)dz = iR\int_0^{2\pi} f\left(Re^{i\theta}\right) e^{i\theta}d\theta.$$

Theorem. *1. (Stoke's Theorem) Let Ω be a bounded domain in $\mathbb{C}$ and $\partial\Omega$ consists of a finite number of C^1 Jordan curves. Let $\omega = Pdx + Qdy$ and $P = P(x,y), Q = Q(x,y)$ are in $C^1\left(\bar{\Omega}\right)$. Then*

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega.$$

Remark: For $\omega = Pdx + Qdy$. Using $(dx \wedge dx = 0)$ and $dx \wedge dy = -dy \wedge dx$ then

$$\begin{aligned} d\omega &= dP \wedge dx + dQ \wedge dy \\ &= (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy \\ &= P_y dy \wedge dx + Q_x dx \wedge dy \\ &= (Q_x - P_y)\, dx \wedge dy. \end{aligned}$$

Theorem. *2. (Complex version) Let Ω be as before. Let $\omega = f(z)dz$, whre $f \in C^1\left(\bar{\Omega}\right)$, complex -valued. Then*

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega = \int_{\Omega} \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$$

Remark: $d\omega = d\left(fdz\right) = df \wedge dz$. (recall $df = f_x dx + f_y dy = f_z dz + f_{\bar{z}} d\bar{z}$.)

Theorem. *3(Cauchy-Green) Let Ω be as in Thereom 2. Then $\forall f \in C^1\left(\bar{\Omega}\right)$, complex-valued. we have*

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi + \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{\xi}} d\xi \wedge d\bar{\xi}}{\xi - z}\right].$$

- Note that Theorem 3 implies Theorem 0, since $\frac{\partial f}{\partial \bar{\xi}} \equiv 0$.

Proof of theorem3:

Proof. Use \zeta instead of \xi. oops. Let $G = \Omega \backslash B_\epsilon(z)$. Then apply Theorem 2 to $\omega = \frac{f(\zeta)}{\zeta - z} d\zeta$. Since $\frac{f(\zeta)}{\zeta - z} \in C^1\left(\bar{G}\right)$, we have by stokes theorem that

$$\int_{\partial G} \omega = \int_G d\omega$$

so

$$\begin{aligned} \int_{\partial G} \omega &= \int_{\partial\Omega} \omega - \int_{\partial B_\epsilon(z)} \omega \\ &= \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial B_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

so that

$$\begin{aligned} \int_G d\omega &= \int_{\Omega\backslash B_\epsilon(z)} d\left(\frac{f}{\zeta - z}\right) \wedge d\zeta \\ &= \int_{\Omega\backslash B_\epsilon(z)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z}. \end{aligned}$$

We claim

(1) $\int_{\partial B_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta \to 2\pi i f(z)$ as $\epsilon \to 0$.

(2) $\int_{\partial\backslash B_\epsilon(z)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z} \to \int_\Omega \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z}$. as $\epsilon \to 0$.

For (1), we put $\zeta = z + \epsilon e^{i\theta}$ for ,$0 \leq \theta < 2\pi$ then

$$\begin{aligned} \int_{\partial B_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta &= (Ex) \quad \int_0^{2\pi} f\left(\epsilon e^{i\theta}\right) d\theta i \\ &\to \quad 2\pi i f(z), \end{aligned}$$

as $\epsilon \to 0$. For (2) we get

$$\begin{aligned} \left| \left(\int_{\partial \setminus B_\epsilon(z)} - \int_\Omega \right) \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z} \right| &= \left| \int_{B_\epsilon(z)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z} \right| \\ &\leq \|f\|_{C^1} \int_{B_\epsilon(z)} \frac{d\bar{\zeta} \wedge \zeta}{\zeta - z} \end{aligned}$$

but $d\zeta \wedge d\bar{\zeta} = -2ir dr \wedge d\theta$ so that $d\zeta = d\left(re^{i\theta}\right) = e^{i\theta} dr + ie^{i\theta} d\theta r$ and $\zeta = z + re\%i\theta$ so that

$$\int_0^\epsilon \int_0^{2\pi} dr d\theta \to 0$$

□

2/12/2015

- Last time:

Theorem. *(Cauchy-Green-Pompei formula). Let Ω be a bounded domain in $\mathbb{C}$ with C^1 boundary $\partial\Omega$. For any $f \in C^1\left(\bar{\Omega}\right)$,*

$$f(z) = f(z) = \frac{1}{2\pi i}\left[\int_{\partial\Omega}\frac{f(\xi)}{\xi - z}d\xi + \int_{\Omega}\frac{\frac{\partial f}{\partial\bar{\xi}}d\xi\wedge d\bar{\xi}}{\xi - z}\right], \quad \forall z\in\Omega.$$

Remark. 1.We have that $i\frac{d\zeta\wedge d\bar{\zeta}}{2}$ is a volume form. Indeed, write $\zeta = re^{i\theta}$. So

$$\begin{aligned} d\zeta &= e^{i\theta}dr + rie^{i\theta}d\theta \\ d\bar{\zeta} &= e^{-i\theta}dr - ire^{-i\theta}d\theta \\ d\zeta\wedge d\bar{\zeta} &= -2irde\wedge d\theta. \end{aligned}$$

Since $dr\wedge dr = d\theta\wedge d\theta = 0$, and $dr\wedge d\theta = -d\theta\wedge dr$. And

$$\begin{aligned} \frac{i}{2}d\zeta\wedge d\bar{\zeta} &= rdr\wedge d\theta \\ \frac{i}{2}\int f\zeta\wedge d\bar{\zeta} &= \int\int f\left(re^{i\theta}\right)rdrd\theta. \end{aligned}$$

Remark. 2. From Last time we proved this by applying stokes theorem to $G = \Omega\backslash B_\epsilon(z)$ Wait

$$\int_{\Omega\backslash B_\epsilon(z)}\frac{\frac{\partial f}{\partial\bar{\xi}}d\xi\wedge d\bar{\xi}}{\xi - z} \to \int_{\Omega}\frac{\frac{\partial f}{\partial\bar{\xi}}d\xi\wedge d\bar{\xi}}{\xi - z}$$

as $\epsilon\to 0$. We estimate

$$\begin{aligned} \left|\int_{\Omega\backslash B_\epsilon}\cdot - \int_{\Omega}\right| &= \left|\int_{B_\epsilon(z)}\frac{\frac{\partial f}{\partial\bar{\xi}}d\xi\wedge d\bar{\xi}}{\xi - z}\right| \\ &= \left|\int\frac{\partial f}{\partial\bar{\xi}}2ie^{-i\theta}drd\theta\right| \\ &\le \|f\|_{C^1}\int\left|2ie^{i\theta}\right|drd\theta \\ &\le 2\|f\|_{C^1}\int_0^\epsilon dr\int_0^{2\pi}d\theta \\ &= 2\|f\|_{C_1}\epsilon 2\pi\to 0 \end{aligned}$$

as $\epsilon\to 0$. with $(\zeta = z + re^{i\theta})$ and $d\zeta = d\left(r^{i\theta}\right)$.

Corollary. *1. (Cauchy Integral formula). Let Ω be a bounded domain in $\mathbb{C}$ with C^1 boundary $\partial\Omega$. Then for all holomorphic f on Ω with $f\in C^1\left(\bar{\Omega}\right)$, then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\xi)}{\xi - z}d\xi, \quad \forall z\in\Omega.$$

Corollary. *2. (Cauchy Integral Theorem). Let Ω be a bounded domain in $\mathbb{C}$ with C^1 boundary $\partial\Omega$. Then if F is holomorphic on Ω and $F\in C^1\left(\bar{\Omega}\right)$, then*

$$\int_{\partial\Omega}F(z)dz = 0.$$

Proof. Apply Corollary 1 to $f(z) = zF(z)$ with $z = 0$. □

Theorem. *1. Let Ω be a bounded domain in $\mathbb{C}$ and $\partial\Omega$consists of a finite number of* <u>*rectifiable*</u> *Jordan curves. Let f be holomorphic on Ω and $f \in C^0\left(\bar{\Omega}\right)$. Then*

$$\int_{\partial\Omega} f(z)dz = 0.$$

($\iff \forall z \in \Omega\ f(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}d\zeta$)

<u>**Corollary 2 $\implies$ Corollary 1:**</u> Apply Stoke's theorem to $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ with $G = \Omega \backslash B_\epsilon(z)$.

Definition. A curve $\gamma : [a, b] \to \mathbb{C}$ is rectifiable if it satisfies the following. Let P be a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$. Denote

$$L(\gamma; P) = \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})|.$$

We say γ is rectifiable if $\sup_P L(\gamma; P) < +\infty$.

Example. 1. A piecewise C^1 curve is rectifiable: Take $\gamma \in C^1$. Then

$$\begin{aligned} L(\gamma; P) &= \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \\ &= \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \gamma'(t)dt \right| \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\gamma'(t)|\, dt \\ &= \int_a^b |\gamma'(t)|\, dt \\ &< \infty. \end{aligned}$$

- In fact, we have that

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Example. 2. Let γ be

$$\gamma(t) = \begin{cases} 0 & t = 0 \\ t + it^\alpha \sin\left(\frac{\pi}{t}\right) & 0 < t \leq 1 \end{cases}$$

where $\alpha > 0$ is a constant. It kinda looking a shrinking topologists sine curve. It can be shown that γ is NOT rectifiable. When $1 < \alpha \leq 2$, we have that γ is NOT piecewise C^1.

Definition. Let γ be a rectifieable curve on $[a, b]$ and f be continuous on $\gamma([a, b])$. Define

$$\int_\gamma f = \int_a^b f(\gamma(t))d\gamma(t) \text{ (Riemman Stieljes).}$$

That is,

$$\text{RHS} = \lim \sum_{j=1}^{m} f(\gamma(c_j))\left[\gamma(t_j) - \gamma(t_{j-1})\right]$$

where $\{a = t_0 < \cdots < t_m = b\}$ is a partition $c_j \in [t_{j-1}, t_j]$.

Example. 1. Want to show that this definition is consistant with our previous definition. That is if $\gamma \in C^1([a, b])$ then

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Proof. We estimate

$$\begin{aligned} \left| \int_\gamma f - \int_a^b f(\gamma(t))\gamma'(t)dt \right| &< 2\epsilon + \left| \sum_{j=1}^m f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{j=1}^m f(\gamma(d_j))\gamma'(d_j)(t_j - t_{j-1}) \right| \\ &< 3\epsilon. \end{aligned}$$

Pick d_j such that $\gamma'(d_j)(t_j - t_{j-1}) = \gamma(t_j) - \gamma(t_{j-1})$ by MVT. □

Example. 2. Compute $\int_\gamma 1, \int_\gamma z$, with γ rectifiable. Note for the first one we compute

$$\sum_{j=1}^m 1(\gamma(t_j) - \gamma(t_{j-1})) = \gamma(b) - \gamma(a),$$

for all partitions P. Thus

$$\int_\gamma 1 = \gamma(b) - \gamma(a).$$

Next, we compute $\int_\gamma z$: Consider the Riemman Stieljes sum

$$\int_\gamma z = \lim \sum_{j=1}^m \gamma(c_j)(\gamma(t_j) - \gamma(t_{j-1})).$$

Put $c_j = t_j$ and ,$c_j = t_{j-1}$ respectively and get

$$\begin{aligned} \int_\gamma z &= \frac{1}{2} \lim \sum_{j=1}^m (\gamma(t_j) + \gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1})) \\ &= \frac{1}{2} \lim \sum_{j=1}^m \left(\gamma(t_j)^2 - \gamma(t_{j-1})^2\right) \\ &= \frac{1}{2}\left(\gamma(b)^2 - \gamma(a)^2\right). \end{aligned}$$

In particular, $\oint_\gamma 1 = \oint_\gamma z dz = 0$.

- For Cauchy Integral Theorem, $\partial\Omega$ rectifieable Jordan $f \in C^0(\Omega)$.
- **Case1:** $\Omega = \mathbb{D} = \{|z| < 1\}$. Assume $f \in C^0(\bar{\Omega}) = C^0(\bar{\mathbb{D}})$, f holomorphic on $\mathbb{D}$. We want

$$\int_{\partial\mathbb{D}} f(z)dz = 0.$$

Proof. Let $\mathbb{D}_r = \{|z| < r\}$. with $0 < r < 1$. By corolary 2, we have

$$\int \partial\mathbb{D}_r f(z)dz = 0$$

which implies

$$\int_0^{2\pi} f\left(re^{i\theta}\right)e^{i\theta}d\theta = 0$$

so that

$$\begin{aligned}\left|\int_{\partial\mathbb{D}} f(z)dz\right| &= \left|\int_0^{2\pi} f\left(e^{i\theta}\right)e^{i\theta}d\theta\right| \\ &= \left|\int_0^{2\pi} f\left(e^{i\theta}\right)e^{i\theta}d\theta - \int_0^{2\pi} f\left(re^{i\theta}\right)d\theta\right| \\ &\leq \int_0^{2\pi}\left|f\left(e^{i\theta}\right) - f\left(re^{i\theta}\right)\right|d\theta \\ &\leq 2\pi\epsilon,\end{aligned}$$

for all $\epsilon > 0$. So that $\int_{\partial\mathbb{D}} f = 0$. □

- **Case2:** (Goursat Theorem) Let Δ be a region of traingle. Assume $\bar{\Delta} \subset \Omega$. f holomorphic on $\bar{\Delta}$. We want

$$\int_{\partial\Delta} f = 0.$$

Proof. Let $I = \int_{\partial\Delta} f$. Then

$$I = \int_{\partial\Delta_1} f + \int_{\partial\Delta_2} f + \int_{\partial\Delta_3} f + \int_{\partial\Delta_4} f$$

and

$$|I| \leq \sum_{n=1}^{4}\left|\int_{\partial\Delta_i} f\right|.$$

There exists one Δ_i such that $\frac{1}{4}|I| \leq \left|\int_{\partial\Delta_i} f\right|$. Denote $\Delta^{(i)} = \Delta_i$. So you have a sequence $\Delta, \Delta^{(1)}, \ldots, \Delta^{(k)}, \cdots$. Then

$$\frac{1}{4^k}|I| \leq \left|\int_{\partial\Delta_k} f\right|, \forall k.$$

Now note that $\bar{\Delta^{(k)}} \subset \bar{\Delta^{(k-1)}}$, with $\mathrm{diam}(\Delta^{(k)}) \to 0$ as $k \to \infty$. By Canto's lemma,

$$\bigcap_{k=1}^{\infty} \overline{\Delta^{(k)}} = \{z_\star\} \subset \Omega.$$

Note that f is holomoprhic on $z_\star$. We have

$$\frac{f(z) - f(z_\star)}{z - z_\star} - f'(z_\star) = R(z_\star, z),$$

where $R(z_\star, z) \to 0$ as $z \to z_\star$. WE have

$$f(z) = f(z_\star) + f'(z_\star)(z - z_\star) + R(z - z_\star)$$

so that

$$\begin{aligned}\int_{\partial\Delta_k} f &= \int_{\partial\Delta_k} f(z_\star)dz + \int_{\partial\Delta_k} f'(z_\star)(z - z_\star)dz + \int_{\partial\Delta_k} R(z - z_\star)dz, \\ &= 0 + 0 + \int_{\partial\Delta_k} R(z - z_\star)dz\end{aligned}$$

wher the first two terms are zero by Example 2. Now we must show the lasdt term is also zero. We have

$$\begin{aligned} \left| \int_{\partial \Delta_k} f \right| &\leq \max |R| \int_{\partial \Delta_k} |z - z_\star| \, |dz| \\ &\leq \max |R| \, \frac{\text{diam}\Delta}{2^k} \frac{L(\Delta)}{2^k} \end{aligned}$$

which implies $|I| \leq \max |R| \, \frac{\text{diam}\Delta}{1} \frac{L(\Delta)}{1} \to 0$ asw $z \to z_\star$. □

- 3 prelims problms, like homoeworks and lecture notes.
- Show a functioin is holomophic, Use $\bar{z}$ method.
  - Make sure you undersand the homeowrk probelms and lecture notes.
  - Read cross material in the books.

2/19/2015

- Last time

Theorem. *0. Let Ω be a bounded domain in $\mathbb{C}$ and $\partial\Omega$consists of a finite number of rectifiable Jordan curves. Let f be holomorphic on Ω and $f \in C^0\left(\bar{\Omega}\right)$. Then*

$$\int_{\partial\Omega} f(z)dz = 0.$$

($\iff \forall z \in \Omega\ f(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{\zeta - z}d\zeta$)

Theorem. *1. Let Ω be a bounded domain in $\mathbb{C}$ and $\partial\Omega$consists of a finite number of rectifiable Jordan curves. Let f be holomorphic on Ω and $f \in C^0\left(\bar{\Omega}\right)$. Then*

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\left(\zeta - z\right)^{k+1}}d\zeta$$

for all $k \geq 0$, all $z \in \Omega$.

Corollary. *2. For every domain $G \subset \mathbb{C}$, not necessarily bounded. If $f'(z)$ exists for all $z \in G$, then $f^{(k)}(z)$ exists for $z \in G$.*

Proof. For all $z \in G$ there exists a ball $B(z, r) \subset G$. Then apply theorem 1 to f and $\Omega = B(z, r)$. □

Proof of Theorem 1:

Proof. For $k = 1$, we have

$$\begin{aligned}
\left|\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\left(\zeta - z\right)^2}d\zeta\right| &= \left|\frac{1}{2\pi i}\frac{1}{h}\int_{\partial\Omega}\left(\frac{f(\zeta)}{(\zeta - z - h)} - \frac{f(\zeta)}{\zeta - z}\right)d\zeta - \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\left(\zeta - z\right)^2}d\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial\Omega} f(\zeta)\left(\frac{1}{\left(\zeta - z - h\right)\left(\zeta - z\right)} - \frac{1}{\left(\zeta - z\right)^2}\right)d\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial\Omega} f(\zeta)\left(\frac{h}{\left(\zeta - z - h\right)\left(\zeta - z\right)}\right)d\zeta\right| \\
&\leq \frac{1}{2\pi}\sup_{\partial\Omega}|f|\cdot|h|\left|\int_{\partial\Omega}\frac{d\zeta}{\left(\zeta - z - h\right)\left(\zeta - z\right)}\right| \qquad (1)
\end{aligned}$$

Now denote $d = \operatorname{dist}\left(z, \partial\Omega\right) > 0$. Then for all h such that $0 < |h| < \frac{d}{4}$. Then

$$|\zeta - z| \geq d$$

so that

$$|\zeta - (z+h)| \geq ||\zeta - z| - |h|| \geq \frac{3d}{4}.$$

But then

$$\left|\int_{\partial\Omega}\frac{d\zeta}{\left(\zeta - z - h\right)\left(\zeta - z\right)}\right| \leq \frac{4}{3d^3}\left|\int_{\partial\Omega} d\zeta\right| \leq \frac{4}{3d^3}L\left(\partial\Omega\right).$$

Plugging this into (1) we get

$$\left|\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\left(\zeta - z\right)^2}d\zeta\right| \leq \frac{1}{2\pi}\sup_{\partial\Omega}|f|\cdot|h|\cdot\frac{4}{3d^3}L\left(\partial\Omega\right) = \frac{2}{3d^3\pi}\sup_{\partial\Omega}|f|\cdot|h|\,L\left(\partial\Omega\right),$$

which proves as needed.

Now for $k \geq 2$, we observe that

$$\frac{1}{h}\left[\frac{(k-1)!}{(\zeta-z-h)^k}-\frac{(k-1)!}{(\zeta-z)^{k+1}}\right]$$

needs to be estimated. There is a little trick (like Abel's theorem). Here it is:

$$(\star)=\frac{1}{h}\left[\frac{(k-1)!}{(\zeta-z-h)^k}-\frac{(k-1)!}{(\zeta-z)^{k+1}}\right] = \frac{(k-1)!}{h}\frac{(\zeta-z)^{k-1}+\cdots+(\zeta-z-h)^{k-1}}{(\zeta-z)^k(\zeta-z-h)^k} -\frac{k!}{(\zeta-z)^{k+1}},$$

where we use $a^k-b^k=(a-b)\left(a^{k-1}+\cdots+b^{k-1}\right)$. Now

$$\begin{aligned}
|(\star)| &= (k-1)!\left|\frac{(\zeta-z)^{k-1}+\cdots+(\zeta-z-h)^{k-1}}{(\zeta-z)^k(\zeta-z-h)^k}-\frac{k}{(\zeta-z)^{k+1}}\right| \\
&\leq (k-1)!\left|\frac{(\zeta-z)\left[(\zeta-z)^{k-1}+\cdots+(\zeta-z-h)^{k-1}\right]-k(\zeta-z-h)^k}{(\zeta-z)^k(\zeta-z-h)^k}\right| \\
&\leq C(k)\,|h|\,d^{-k-2}\to 0
\end{aligned}$$

as $h\to 0$ we're we used a estimate similar to the $k=1$ case. □

Corollary. *3. Let Ω be a domain in $\mathbb{C}$. Suppose $\overline{B(z,R)}\subset\Omega$. Then*

$$\begin{aligned}
\left|f^{(k)}(z)\right| &\leq \frac{k!}{R^k}\sup_{\partial B(z,R)}|f| \\
&\leq \frac{k!}{R^k}\sup_{\overline{B(z,R)}}|f|.
\end{aligned}$$

Proof. Apply Theorem 1 to f and $\Omega=B(z,R)$. Then

$$\begin{aligned}
\left|f^{(k)}(z)\right| &= \left|\frac{k!}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta-z)^{k+1}}d\zeta\right| \\
&= \frac{k!}{2\pi}\left|\int_{\partial B}\frac{f(\zeta)}{(\zeta-z)^{k+1}}d\zeta\right| \\
&= \frac{k!}{2\pi}\sup_{\partial B}|f|\left|\int_{\partial B}\frac{d\zeta}{(\zeta-z)^{k+1}}\right| \\
&\leq \frac{k!}{2\pi}\frac{L(\partial B)}{R^{k+1}}\sup_{\partial B}|f| \\
&= \frac{k!}{R^k}\sup_{\partial B}|f|.
\end{aligned}$$

□

Corollary. *4. Let Ω be a domain in $\mathbb{C}$. Let $K\subset\Omega$ be a compact set. Then for any neighborhood V of K in Ω,*

$$\sup_{z\in K}\left|f^{(k)}(z)\right|\leq C\sup_V|f|,$$

whre $C > 0$ is a constant depending only on K, $dist(K, \partial V)$.

Remark. A neighborhood V of K is called rel compact if $\overline{V}$ is compact. So the corol in Corollary 4. We have $K \subset V \subset \overline{V} \subset \Omega$.

Proof 1 of Corolary4:

Proof. For all $z \in K$, pick $\overline{B(z, R_z)} \subset V$. So that

$$K \subset \bigcup_{i=1}^{N} \overline{B(z, R_z)}.$$

Cor3! □

Theorem. *(Liouville's Theorem) Let f be a holomorphic funciton on the whole plane $\mathbb{C}$. If f is bounded on $\mathbb{C}$, then $f \equiv$ constant.*

Proof1:

Proof. For all $z \in \mathbb{C}$ by Coro3.(Cauchy's estimate) Then

$$|f'(z)| \leq \frac{1}{R} \sup_{\partial B(z,R)} |f|.$$

Note that this holds for all $R > 0$. So that

$$\sup_{\partial B} |f| \leq \sup_{\mathbb{C}} |f| = M.$$

This means that

$$|f'(z)| \leq \frac{M}{R} \to 0$$

as $R \to +\infty$. Hence $|f'(z)| = 0$ which implies $f'(z) \equiv 0$ for all $z \in \mathbb{C}$. This shows f mus the constant. □

- The key is to control $|\nabla f|$ by $|f|_{C^0}$.

Theorem. *3. (Morera) Let f be a continuous function on a domain Ω. If $\int_\gamma f = 0$ for any closed rectifiable curve inside Ω, then f is holomorphic on Ω.*

Proof. Fix $z_0 \in \Omega$. For all $z \in \Omega$. Pick a rectifiable curve γ_1 joining z_0 and z_1. Define a function

$$F(z) = \int_{\gamma_1} f = \int_{z_0}^{z} f.$$

Check that $\int_{z_0}^{z} f$ is well-defined. Then we can show

$$F'(z) = f(z)$$

by the homework problem. And proceed by showing that f must have a primitive F. Since the primitive F is holomorphic then f must be holomorphic. □

Theorem. *4. (Series representation) Let Ω be a domain and $B(z_0, R) \subset \Omega$. Let f be holomorphic on Ω. Then*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \forall z \in B(z_0, R)$$

where the series RHS converges absolutely and uniformly on $\overline{B(z_0,R)}$ *and*

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta$$

for $k \geq 0$*.*

- Sometimes this is used as the definition of a holomorphic function.
- We need a lemma to prove this theorem.

Lemma. *1. Let* γ *be a rectifiable curve. Let* $\{F_n\}$ *be a sequence of continuous functions on* γ*. Suppose* F_n *converges uniformly to* F *. Then*

$$\lim_{n\to\infty}\int_\gamma F_n = \int_\gamma f.$$

Proof. By the linearity we have

$$\begin{aligned}\left|\int_\gamma F_n - \int_\gamma f\right| &= \int_\gamma |F_n - F| \\ &< \epsilon\left|\int_\gamma 1\right| \\ &\leq \epsilon L(\gamma) \to 0\end{aligned}$$

as $\epsilon \to 0$, because the convergence is uniform. □

Proof of Theorem 4:

Proof. For all $z \in B$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

Then

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0}\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} \\ &= \frac{1}{\zeta - z_0}\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(\zeta-z_0)^k}\end{aligned}$$

if $|z-z_0| < |\zeta - z_0| = R$. Then use the lemma to prove the rest. Left as an exercise to the reader. □

2/24/2015

- Midterm 1 Review:
  - We have that $\Delta |f|^2 = 4\left|\frac{\partial f}{\partial z}\right|^2$. This implies $|f|^2$ is subharmonic.
  - Proof: $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.
  - Then

$$\begin{aligned}\Delta |f|^2 &= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}} f\bar{f}\right] = 4\frac{\partial}{\partial z}\left[f\frac{\partial \bar{f}}{\partial \bar{z}}\right], \ \frac{\partial}{\partial \bar{z}} f = 0 \\ &= 4\frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} \\ &= 4\left|\frac{\partial f}{\partial z}\right|^2.\end{aligned}$$

  - For $p > 0$ we have

$$\begin{aligned}\Delta\left(|f|^p\right) &= \Delta\left(|f|^2\right)^{p/2} \\ &= 4\frac{\partial^2}{\partial z\partial \bar{z}}\left(|f|^2\right)^{p/2} \\ &= 4\frac{\partial}{\partial z}\left[\left(\frac{p}{2}\right)\left(|f|^2\right)^{\frac{p}{2}-1} f\frac{\partial \bar{f}}{\partial \bar{z}}\right] \\ &= 4\left(\frac{p}{2}\right)\left(|f|^2\right)^{\frac{p}{2}-1}\left|\frac{\partial f}{\partial z}\right|^2 \\ &\quad +4\left(\frac{p}{2}\right)\left(\frac{p}{2}-1\right)\left(|f|^2\right)^{\frac{p}{2}-2}\bar{f}\frac{\partial f}{\partial z}\cdot f\cdot\frac{\partial \bar{f}}{\partial \bar{z}} \\ &= 4\left(\frac{p}{2}\right)\left(|f|^{p-2}\right)\left|\frac{\partial f}{\partial z}\right|^2 \\ &\quad +4\left(\frac{p}{2}\right)\left(\frac{p}{2}-1\right)\left(|f|^{p-4}\right)|f|^2\left|\frac{\partial f}{\partial z}\right|^2 \\ &= p^2|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2\end{aligned}$$

  - Remark: If $|f|^p = f^{\frac{p}{2}}\bar{f}^{\frac{p}{2}}$ then we need to worry about the branch of $f^{\frac{p}{2}}$, p not even.
- #2: $\frac{dz}{z} = id\theta$.
- #3. Two methods:
  - Method 1. Suppose that $u = \log|z|$ has a harmonic conjugate v. Then $f = u+iv$ is holomorphiv on $\Omega \supset \{|z| = 1\}$. Then use

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

    Using these equations we get that we must have $v = \theta + C$ where C is a real constant. So $f = u + i(\theta + C)$ on Ω. But then $e^f = ze^{ic}$ so $f - ic$ is a branch of $\log z$ on Ω. But $\log z$ does not have a branch on Ω. Contradiction!

– Method2: Let $f' = u_x + iv_x = u_x - iu_y = \frac{x-iy}{x^2+y^2} = \frac{1}{z}$ on Ω. Then

$$\int_\gamma f'dz = f(\gamma(1)) - f(\gamma(0)) = 0$$

since γ is a closed curve. On the other hand,

$$\int_\gamma f'dz = \int_\gamma \frac{1}{z}dz = 2\pi i.$$

By either Cauchy's formula. Apply to $f(\zeta) = 1$, or simply change coordinates and compute

$$\int_\gamma f'dz = \int_0^{2\pi} id\theta = 2\pi i.$$

In either case we get a contradiction.

Homework Problem:

Exercise. (August 2012) Suppose the function f is analytic in a simply-connected domain Ω. Show that there exists an analytic function F in Ω such that $F'(z) = f(z)$ for all $z \in \Omega$.

- Remark: We cannot define $F(z) = \int_{z_0}^{z} f(z)dz$ for a general holomorphic f on Ω.
- For example if we have a hole, then this function will not be well defined.
- However, if Ω is simple connected, then F is well defined. So $\gamma_1 - \gamma_2$ ecnlosed a domain inside Ω. So

$$\int_{\gamma_1-\gamma_2} fdz = 0.$$

Important Theorems for this week:

- Rouche's Theorem
- Maximum Principle
- Schwarz lemma.

Zeros:

- Let f be a holomorphic function on domain Ω. A point $z_0 \in \Omega$ is a **zero** of f if $f(z_0) = 0$.
- If $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, and $f^{(m)}(z_0) \neq 0$. Then z_0 is a zero of f with **multiplicity** m. If $m = 1$, z_0 is a **simple zero of** f

Lemma. *1. Let f be a holomorphic function on Ω. a point z_0 is a zero of f with multiplicity m if and only if there exists a disk $B(z_0, \delta)$ centereds at z_0 of radius $\delta > 0$, such that*

$$f(z) = (z - z_0)^m h(z)$$

where h is holomorphic on $B(z_0, \delta)$ and $h \neq 0$.

Proof. ($\Longleftarrow$)
Trivial
($\Longrightarrow$)
By the series representation of f there exists a $B(z_0, R)$ such that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

on $\overline{B(z,R)}$ with $a_j = \frac{f^{(j)}(z_0)}{j!}$. Then $a_0 = \cdots = a_{m-1} = 0$, so that

$$f(z) = (z - z_0)^m \sum_{j=0}^{\infty} a_{m+j} (z - z_0)^j .$$

Here this new series $\sum_{j=0}^{\infty} a_{m+j} (z - z_0)^j$ and $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ converge for the same z. So $\sum a_{m+j} (z - z_0)^j$ defined a holomorphic function h. Now $h(z_0) = a_m \neq 0$. By continuity, there exists $\delta \in (0, R)$ such that $h(z) \neq 0$, for all $z \in B(z_0, \delta)$. Then $f = (z - z_0)^m h$ on $B(z_0, \delta)$. □

Theorem. *2. Let f be a holomorphic on Ω. Then the set of zero of f has no limit points in Ω, unless f is identically zero.*

Corollary. *3. Let f_1 and f_2 be two holomorphic on Ω. If there exists a set $E \subset \Omega$ such that E contains a limit point of Ω and $f_1 = f_2$ on E, then $f_1 = f_2$ on Ω.*

Proof. Theorem 2 implies Coro 3. Apply Theorem 2 to $f = f_1 - f_2$. □

- Applications: $\sin^2 z + \cos^2 z = 1$ $(\star)$. Since $\sin z, \cos z$ are holomorphic, when z is real then $(\star)$ holds. Let E =real axis.

Proof of Theorem 2

Proof. Let $U = \left\{z \in \Omega \mid \exists \overline{B(x,\delta)} \subset \Omega \text{ s.t.} f \equiv 0 \text{ on } B(z,\delta)\right\}$. Also let $V = \left\{z \in \Omega \mid \exists \overline{B(x,\delta)} \subset \Omega \text{ s.t.} f \neq 0 \text{ on } B(z,\delta) \backslash \{z\}\right\}$.

We claim: $\Omega = U \bigsqcup V$.

Now clearly $U \cap V = \emptyset$. Also U, V are open. If $f \neq 0$ on Ω, then $\Omega \subset V$. If there eixsts z_0 is a zero of multiplicity m, then by Lemma, $z_0 \in V$. If there exists z_1 such that $f^{(k)}(z_1) = 0$ for all $k \geq 1$. By series represeation $f \equiv 0$ on some $B(z_1, \epsilon_1)$ so $z_1 \in U$. So $\Omega \subset U \bigsqcup V \subset \Omega$. The claim is proved. Now suppose $z_\star$ is a limit poibt of $Z(f)$ =set of zeros of f. There exists a sequence z_k of distinct points $z_k \to z_\star$. Then $f(z_\star) = \lim_{k\to\infty} f(z_k) = 0$. Suppose $z_\star$ has multiplicity m then $f(z) = (z - z_\star)^m h$. on $B(z_\star, \delta)$., $h \neq 0$. But when k is sufficiently large, we have

$$0 = f(z_k) = (z_k - z_\star)^m h(z_k).$$

So $f^{(k)}(z_\star) = 0$, for all k. Hence, $z_\star \in U$, Hence $V = \emptyset$. Now $\Omega = U$. □

- Example: $Z(f)$ could have a limit point on $\partial\Omega$. Consider $f(z) = \sin \frac{1}{1-z}$ and take $z_k = 1 - \frac{1}{2k\pi}$. but $1 \in \partial\mathbb{D}$.

Theorem. *3. (Argument Principle) Let f be a holomorphic on Ω. Suppose there is a rectifiable Jordan curve $\lambda \subset \Omega$ such that $f\mid_\gamma \neq 0$ and γ encloses a domain U in Ω. Then the number of zeros of $f$in U is given by*

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}.$$

Theorem. *4. (Rouche's Theorem) Let Ω be a domain and $\gamma \subset \Omega$ resdtifiable Jordan curve encloses a (bounded) domain U in Ω. Suppose f and g are holo on Ω and*

$$|g(z)| < |f(z)|$$

on γ. Then f and $f \pm g$ have the same number of zeros counting multiplicities.

Application:

Theorem. *(Fundamental Theorem of Algebra) Any kth degree polynomial has exactly k roots with multiplcities.*

Proof. Let $P(z) = a_k z^k + \cdots + a_1 z_1 + a_0$. Pick $g(z) = a_{k-1} z^{k-1} + \cdots + a_0$. Then

$$|g(z)| < |P(z)|$$

when $|z| = R$. So $P(z)$ and $P(z) - g(z) = a_k z^k$ have k roots (mult) on $B(0, R)$. □

2/26/2015

Lemma. *Let f be a holomorphic function on Ω. a point z_0 is a zero of f with multiplicity m if and only if there exists a disk $B(z_0, \delta)$ centereds at z_0 of radius $\delta > 0$, such that*

$$f(z) = (z - z_0)^m \, h(z)$$

where h is holomorphic on $B(z_0, \delta)$ and $h \neq 0$.

Theorem. *Let f be holomorphic on Ω. We have $f \equiv 0$ on Ω if one of two properties holds:*
(i) $\exists z_0 \in \Omega$ such that $f^{(k)}(z_0) \equiv 0$ for all $k \geq 0$.
(ii) The zero set $Z(f)$ has a limit point in Ω.

Corollary. *(Holo continuation)*

Theorem. *1 (Arg. Principle) Let f be holomorphic on Ω. Let $\gamma \subset \Omega$ be rectifiable Jordan curve which enclosed a bounded domain $U \subset \Omega$. Assume $f \mid_\gamma \neq 0$. Then the number of zeros of f in U counting multiplicities is*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}.$$

Proof. First, f has only finitely many zeros in U. For, otherwise $Z(f)$ has a limit point (because every infinite set has a limit point) , which implies $f \mid_\Omega = 0$, a contradiction.

Second, let $\{z_1, \ldots, z_p\}$ be the zero set of f in U. There exists $\overline{B(z_i, \delta_i)} \subset U$ where $f(z) = (z - z_i)^{m_i} \, h_i(z)$, with $i = 1, \ldots, p$. Denote $\gamma_i = \partial B(z_i, \delta_i)$. Apply Cauchy integral Theorem to $\frac{f'}{f}$ on $U \setminus \cup_1^p B(z_i, \delta_i)$. Thus

$$0 = \int_{\gamma - \bigcup_{i=1}^p \gamma_i} \frac{f'}{f},$$

which implies

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = \frac{1}{2\pi i} \sum_{j=1}^p \int_{\gamma_j} \frac{f'}{f}.$$

For each $j = 1, \ldots, p$ we have

$$\begin{aligned} \int_{\gamma_j} \frac{f'}{f} &= \int_{\gamma_j} \frac{m_i}{z - z_i} + \int_{\gamma_j} \frac{h_i'(z)}{h_i(z)} \\ &= \int_{\gamma_j} \frac{m_i}{z - z_i} + 0, \text{ since } \frac{h_i'(z)}{h_i(z)} \text{ is holomorphic} \\ &= 2\pi i m_j, \end{aligned}$$

as neeeded. □

- Remark: Note $\frac{f'}{f}$" = " $(\log f)'$. (Need to check $f(\gamma) \subset \mathbb{C} \backslash (-\infty, 0]$)
- so that $\oint_\gamma \frac{f'}{f}$" = " $\oint_\gamma \arg f d\theta$ where $\log f = \log |f| + i \arg f$.

Theorem. *2. (Rouche') Let f, g be holomorphic on Ω. Let $\gamma \subset \Omega$ be rectifiable Jordan curve. Assume*

$$|g| < |f| \text{ on } \gamma. \quad , (\star)$$

Then $\#Z(f) = \#Z(f \pm g)$, counting multiplicities in U.

Proof. Only show $\#Z(f)=\#Z(f+g)$. Let $F=f+g$. By arg. principle. We have

$$\#Z(f)=\frac{1}{2\pi i}\int_\gamma\frac{f'}{f}\text{ and }\#Z(F)=\frac{1}{2\pi i}\int_\gamma\frac{F'}{f}$$

so that

$$\begin{aligned}\#Z(f)-\#Z(F) &= \frac{1}{2\pi i}\int_\gamma\left(\frac{f'}{f}-\frac{F'}{F}\right)\\ &= \frac{1}{2\pi i}\int_\gamma\frac{f'F-fF'}{fF}\\ &= \frac{1}{2\pi i}\int_\gamma\frac{\frac{(f'F)}{f^2}-\frac{fF'}{f^2}}{\frac{F}{f}}\\ &= \frac{-1}{2\pi i}\int_\gamma\frac{(F/f)'}{(F/f)}\\ &= (\star)\end{aligned}$$

We claim

$$\frac{(F/f)'}{(F/f)}=\left(\log\frac{F}{f}\right)'\text{ on }\gamma.$$

We show the inequality assumption to show the claim. To see the claim we need to check that this holds (according to the remark):

$$\left(\frac{F}{f}\right)(\gamma)\subset\mathbb{C}\backslash(-\infty,0].$$

In fact,

$$\left|\frac{F}{f}\right|=\left|\frac{f+g}{f}\right|=\left|\frac{g}{f}+1\right|,$$

so that

$$\left|\frac{F}{f}-1\right|=\left|\frac{g}{f}\right|<1\text{ on }\gamma,$$

by the original assumption. Then

$$\frac{F}{f}(\gamma)\subset B(1,1)\subset\mathbb{C}\backslash(-\infty,0].$$

Take the principal branch of log,so that $\log\frac{F}{f}$ is holomorphic on γ. Thus

$$\#Z(f)-\#Z(F)=(\star)=\frac{-1}{2\pi i}\int_\gamma\left(\log\frac{F}{f}\right)'=0,$$

by the Fundamental Theorem of Line integrals. Thus we need this assumption to make sense of the log. □

- You can't directly apply to homeowork problem 2. But you can find large ball. Semi circle. Show $\left|z^3+z+1\right|\sim R^3$ in the circle part, which is why you only have to worry about the imaginary part.
- Remark: Rouche' Theorem can be slightly improved. Condition $(\star)$ cna be relaxed to

$$|g|<|f|+|f\pm g|\text{ on }\gamma.$$

  - But this is not important for us.
  - The first version is good enough for the prelim. In practice, you just find a closed curve, usually a straight line togerther with a circle.

- Here's another application of Rouche's

Theorem. *(Open mapping Theorem) Let f be holo on Ω. Then for all $z_0 \in \Omega$, let $w_0 = f(z_0)$. Assume that z_0 is a zero of $f(z) - w_0$ with multiplicity m. Then for all small $p > 0$, there exists $\delta > 0$ such that $\forall w \in B(w_0, \delta)$, we have $f(z) - w$, has m zeros in $B(z_0, \rho)$. In particular, $B(z_0, \delta) \subset f(\Omega)$. That is, f is open.*

Proof. There exists $\overline{B(z_0, R)} \subset \Omega$ such that $f(z) - w_0 = (z - z_0)^m h(z)$ where $h|_{\overline{B(z_0,R)}} \neq 0$. On $\partial B(z_0, R)$, we have

$$\begin{aligned} |f(z) - w_0| &= R^m |h(z)| \\ &\geq R^m \min_{z \in \partial B} |h(z)| . \end{aligned}$$

Pick $\delta = R^m \min_{z \in \partial B} |h(z)| > 0$. Then for any $w_1 \in B(w_0, \delta)$. we have

$$|w_1 - w_0| < \delta < |f(z) - w_0|$$

on $\partial B(z_0, R)$. Then by Rouche's Theorem,

$$\begin{aligned} m &= \#(f - w_0) \text{ in } B(z_0, R) \\ &= \#(f - w_0 - w_1 + w_0) \text{ in } B(z_0, R) \\ &= \#(f - w_1) \text{ in } B(z_0, R), \end{aligned}$$

which completes the proof. □

Maximum Principle:

Lemma. *(Mean Value Equality) Let f be holomorphic on Ω. Then for all $\overline{B(z, \delta)} \subset \Omega$. Then*

$$f(x) = \fint_{\partial B(x,R)} f(\zeta) d\zeta.$$

and $\fint_{\partial B(x,R)} \cdot = \frac{1}{2\pi R} \int_{\partial B(x,R)} \cdot$.

Proof. By the Cauchy integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(x,R)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

let $\zeta = z + Re^{i\theta}$. Then (since $d \log$ is well defined)

$$\frac{d\zeta}{\zeta - z} (= d \log(\zeta - z)) = i d\theta.$$

so that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B(x,R)} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi R} \int_0^{2\pi} f\left(z + Re^{i\theta}\right) R d\theta \\ &= \fint_{\partial B(x,R)} f d\zeta. \end{aligned}$$

□

Theorem. *(Maximum Principle) Let f be holomorphic on Ω. If there exists $z_0 \in \Omega$ such that*
$$|f(z_0)| \geq |f(z)|, \forall z \in \Omega.$$
then $f \equiv const$ on Ω.

- One of the homework problems needs maximum principle.

Proof. Let
$$U = \{z \in \Omega \mid |f(z)| = |f(z_0)|\}.$$
Since $z_0 \in U$, $U \neq \emptyset$. Now U is closed since $|f|$ is continuous.

Claim: U is open.

Assume the claim then $\Omega = U$. That is, $\forall z \in \Omega$, we have $|f(z)| \equiv$ const. By Homework 1, we have $f \equiv$ const. It remains to show the claim.

Proof of Claim:

For all $z \in U$, there exists $\overline{B(z,\delta)} \subset \Omega$. Suppose there exists $z_1 \in B(z,\delta)$ such that
$$|f(z_1)| < |f(z)| = |f(z_0)|.$$
Pick $\delta_1 = |z - z_1|$. By MVE we have
$$\begin{aligned} |f(z)| &= \left| \frac{1}{2\pi\delta_1} \int_{\partial B(z,\delta_1)} f(\zeta) d\zeta \right| \\ &\leq \frac{1}{2\pi\delta_1} \int_{\partial B(z,\delta_1)} |f(\zeta)|\, d\zeta \\ &< |f(z_0)| \\ &= |f(z).|, \end{aligned}$$
a contradiction. Hence there can't exists a $z_1 \in z_1 \in B(z,\delta)$ with $|f(z_1)| < |f(z)| = |f(z_0)|$, so that $|f(z_1)| = |f(z_0)|$ for all $z_1 \in B(z_1,\delta)$. Thus $B(z,\delta) \subset U$. Thi shows U is open. $\square$

Corollary. *(IMPORTANT)Let f be holomorphic on a bounded domain Ω. Then $\max_\Omega |f|$ is attained only on $\partial\Omega$ unless $f \equiv cosnt$.*

Remark. By switching to the reciprocal, we can get the minimum modulus principle. It states that if f is holomorphic within a bounded domain D, continuous up to the boundary of D, and non-zero at all points, then $|f(z)|$ takes its minimum value on the boundary of D.

Theorem. *5. (Schwarz Lemma) Let $\mathbb{D} = B(0,1)$ (open). Let ,$f : \mathbb{D} \to \mathbb{D}$ holomorphic with $f(0) = 0$. Then*
$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$
In addition, if $|f'(0)| = 1$, or $|f(z)| = |z|$ for some $z \neq 0$ with $z \in \mathbb{D}$, then $f(z) = e^{ia}z$, where $a \in \mathbb{R}$, constant.

Proof. Let
$$F(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$
Then F is holomorphic on $\mathbb{D}$. For by the lemma, we can write $f(z) = z^m h(z)$ on $B(z,\epsilon)$ for small ϵ, where $h\mid_{B(z,\epsilon)} \neq 0$, $m \geq 1$. With
$$\frac{f(z)}{z} = z^{m-1}h(z)$$

holomorphic. Tricky here is that we don't know if this is continuous on boundary or not. Then note that

$$\max_{|z|\leq 1-\epsilon}|F(z)| \leq \frac{\max|f|}{1-\epsilon} \leq \frac{1}{1-\epsilon}, \forall \epsilon > 0.$$

So that by the corollary

$$\sup_{\mathbb{D}}|F(z)| < \frac{1}{1-\epsilon}, \forall \epsilon > 0.$$

Letting $\epsilon \to 0$ we have that

$$\sup_{\mathbb{D}}|F(z)| \leq 1$$

which implies

$$|f(z)| \leq |z|.$$

□

3/3/2015

- Last time

Theorem. *(Maximum Principle)Let* Ω *be a bounded domain and* f *holo on* Ω *and* $f \in C^0\left(\overline{\Omega}\right)$. *Assume* $f\mid_{\Omega} \neq$ *const. If* $|f(z_1)| = \max_{\overline{\Omega}} |f(z)|$ *then* $z_1 \in \partial\Omega$.

Theorem. *0 (Schwarz Lemma) Let* $\mathbb{D} = B(0,1)$ *(open). Let* $f : \mathbb{D} \to \mathbb{D}$ *holomorphic with* $f(0) = 0$. *Then*

$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$

In addition, if $|f(z_1)| = |z_1|$ *for some* $z \neq 0$ *with* $z_1 \in \mathbb{D}$ *or* $|f'(0)| = 1$ *then* $f(z) = e^{ia}z$, *where* $a \in \mathbb{R}$, *constant.*

Proof. Let

$$F(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0. \end{cases}$$

Then F is holomorphic on $\mathbb{D}$. Fix any $z_1 \in \mathbb{D}\setminus\{0\}$. For all $0 < \epsilon < \frac{1}{2}(1 - |z_1|)$, Apply Max Principle to $|z| \leq 1 - \epsilon$. Thus

$$\begin{aligned} |F(z_1)| &\leq \max_{\{|z| \leq 1-\epsilon\}} |F(z)| \\ &\leq \max_{\{|z|=1-\epsilon\}} |F(z)| \\ &= \frac{\max_{\{|z|=1-\epsilon\}} |f(z)|}{1-\epsilon} \\ &< \frac{1}{1-\epsilon}. \end{aligned}$$

Letting $\epsilon \to 0$, we have that $|F(z_1)| \leq 1$, for all $z_1 \in \mathbb{D}$. But this implies that

$$|f(z)| \leq |z| \text{ or } |f'(0)| \leq 1.$$

Now if $|f(z_2)| = |z_2|$ then this implies that $|F(z_2)| = 1$. And if $|f'(0)| = 1$ then this implies that $|F(0)| = 1$.

Now locally we can write $f(z) = z^m h(z)$ for some holo h so that $\frac{f(z)}{z} = z^{m-1}h(z) \to 0$.

By Strong Max principlex, we have that $F(z) \equiv c$ with $|c| = 1$ and so $|F(z)| \equiv 1$. Thus $f(z) = cz$ which implies $f(z) = e^{i\alpha}z$, with $c = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. □

Definition. Let $\Omega \subset \mathbb{C}$ as domain. Define

$$\text{Aut}(\Omega) = \{f : \Omega \to \Omega \text{ biholo}\},$$

where biholomorphic means f^{-1} exists and is holomorphic.

Claim. We claim that

$$\text{Aut}(\Omega) = \{f : \Omega \to \Omega \text{ holo,1-1,and onto}\}$$

Proof. If $f : \Omega \to \Omega$ is 1-1, then $f'(z) \neq 0$ for all $z \in \Omega$. (For if $f'(z_0) = 0$, then $f - f(z_0) = (z - z_0)^m h(z)$ with $m > 1$) By IVT, f^{-1} is holo. □

- **<u>Goal:</u>** Determine $\text{Aut}(\mathbb{D})$.

Proposition. *1.* $\left\{\psi_\alpha(z) = e^{i\alpha}z\right\}_{\alpha\in\mathbb{R}}$ *forms a subgroup of* $Aut(\mathbb{D})$.

Proposition. *2. For any* $a \in \mathbb{D}$, *let* $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, *for all* $|z| < 1$. *Then* $\varphi_a \in Aut(\mathbb{D})$.

Proof. By HW1, we have that $\varphi_a(\mathbb{D}) \subset \mathbb{D}$. Also $\frac{\partial \varphi_a}{\partial \bar{z}} = 0$ which implies φ_a is holomorphic on $\mathbb{D}$. Can solve $\varphi_a^{-1} = \varphi_{-a}$ holomorphic. □

Lemma. *If $f \in Aut(\mathbb{D})$ with $f(0) = 0$, then $f(z) = e^{i\alpha}z$.*

Proof. By Schwarz Lemma, we have that $|f'(0)| \leq 1$. Apply Scwarz Lemma to f^{-1} and get that

$$\frac{1}{|f'(0)|} = \left|\left(f^{-1}\right)'(0)\right| \leq 1$$

So that $f'(0) = 1$ so that $f(z) = e^{i\alpha}z$ as needed. □

Theorem. *2. For all $f \in Aut(\mathbb{D})$ we have that*

$$f = \varphi_a \circ \psi_\alpha = \psi_\beta \circ \varphi_b.$$

Proof. Let $a = f(0)$. Consider $g(z) = (\varphi_a \circ f)(z)$. (standard trick, use composition of Mobious transform.) Now $g : \mathbb{D} \to \mathbb{D}$ and $g \in \mathrm{Aut}\mathbb{D}$. Then $g(0) = 0$ which impliez $g = \psi_\alpha$ by Lemma. □

- Question: What is $\mathrm{Aut}(\mathbb{D}^\star)$ where $\mathbb{D}^\star$ is the punctured disk, that is $\mathbb{D}^\star = \mathbb{D}\setminus\{0\}$?
- Read: The Schwarz-Pick lemma. Weierstrrass Theory,

Theorem. *3. Let $\Omega \subset \mathbb{D}$ be a domain and $\{f_k\}$ be a sequence holomorphic on Ω, such that f_k converges to f uniformly on every compact set of Ω. (This is W-Weisrstrass assumption) Then f is holo on Ω, and $\{f_k'\}$ converges to f' uniformly on every compact set of Ω.*

- Not true in general in Real Analysis.

Proof. By Morera's Theorem for holomorpgic functions, it suffices to show

$$\int_\gamma f = 0$$

for all closed rectfiable curve $\gamma \subset \Omega$. Since $f_n \rightrightarrows f$ on γ. Then

$$\int_\gamma f = \lim_{k\to+\infty} \int_\gamma f_k = 0.$$

For all $K \subset \Omega$ compact, and for all $z \in K$ we have that

$$f_k'(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f_k(\zeta)}{(\zeta - z)^2} d\zeta$$

where B is a large open set in Ω with $B \supset K$. Now

$$f'(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

Now

$$\begin{aligned} |f_k'(z) - f'(z)| &\leq \frac{1}{2\pi}\left|\int_{\partial B} \frac{|f_k(\zeta) - f(\zeta)|}{(\zeta - z)^2} d\zeta\right| \\ &\leq \sup_{z\in K} |f_k - f| \frac{C}{\mathrm{diam} B} \\ &\to 0 \end{aligned}$$

as $k \to +\infty$. □

Corollary. *With (W), $f_k'' \rightrightarrows_{cpt} f''$ and $f_k^{(l)} \rightrightarrows_{cpt} f^{(l)}$.*

Lemma. *2. (Hurwitz) With assumption (W), Let $\gamma \subset \Omega$ be rectifiable cruve which enclosed a bounded domain U in Ω. Assume $f\mid_{\gamma} \neq 0$. Then $\#Z_U(f) = \#Z_U(f_k)$ for all sufficiently large k.*

Proof. By Weiestrass Theorem, f is holomorphic. Since $f\mid_{\gamma} \neq 0$, pick $\delta = \min_{\gamma} |f| > 0$. Now we want to apply Rouche's theorem: We want

$$|f - f_k| < \frac{\delta}{2} < |f| \text{ on } \gamma$$

for k sufficienitly large. By Roche's Theorem, we have that $\#Z_U(f) = \#Z_U(f_k)$. □

Theorem. *(Hurwitz) With (W) if f_k never vanishes on Ω, then either f never vanishes on Ω or , $f \equiv 0$ on Ω.*

Proof. Assume $f(z_0) = 0$ and $f \not\equiv 0$ on Ω. Then $f(z) = (z - z_0)^m h(z)$ on $B(z_0, 2\delta)$ which implies $f\mid_{\partial B(z_0,\delta)} \neq 0$. Applying the previous lemma to $\gamma = \partial B(z_0, \delta)$ yields $f_k(z_0) = 0$ for large k. This is a contradiction. □

Theorem. *(Univalent) With (W) if each f_k is 1-1, then f is 1-1 unless $f \equiv const.$*

Proof. Assume $f(z_1) = f(z_2) = w_0$ for some $z_1 \neq z_2$ in Ω. Consider the function $f(z) - w_0$ which never vanishes on $\partial B(z_1, \delta_1)$ and $\partial B(z_2, \delta_2)$. (Assume $f \not\equiv c$). Here $\delta_{1.}$ and δ_2 are small such that

$$\partial B(z_1, \delta_1) \cap \partial B(z_2, \delta_2) = \emptyset.$$

Now apply Hurwitz lemma to $\gamma_1 = \partial B(z_1, \delta_1)$ and $\gamma_2 = \partial B(z_2, \delta_2)$. Then

$$f_k(z_1') - w_0 = 0 = f_k(z_2') - w_0$$

which implies $f_k(z_1') = f_k(z_2')$ where $z_i' \in B(z_i, \delta_i)$. This is a contradiction of the univalent theorem. □

3/5/2015

- Last time:

Theorem. *(Weirstrass) If* $\sum_{k=1}^{\infty} f_k \rightrightarrows_{cpt} f$ *in* Ω*, then* f *is holo on* Ω *and*$\sum_{k=1}^{\infty} f_k' \rightrightarrows f'$*. (cpt means convergence pointwise? NO)*

Laurent Series

- A general series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n \equiv f(z)$$

is called a **Laurent series centered at** $a \in \mathbb{C}$ where $c_n \in \mathbb{C}$,for all $n \in \mathbb{Z}$. We can write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n (z-a)^m + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n} \\ &= f_H(z) + f_P(z). \end{aligned}$$

- Here $f_P(z)$ is called the principal part of f, and $f_H(x)$ is called the holomorphic part of f.
- We say a **Laurent series** is convergent at z_0 is both f_P and f_H converge at $z = z_0$.
- For f_H, note that

$$\frac{1}{R} = \limsup_{n\to+\infty} \sqrt[n]{|c_n|}$$

is radius of convergence. So f_H converges on $B(a,R) = \{|z-a| < R\}$ and f_H converges uniformly on $B(a,p)$ for all $0 < \rho < R$. For f_P, let $w = (z-a)^{-1}$. Then

$$f_P = \sum_{n=1}^{\infty} c_{-n} w^n.$$

Let

$$r = \limsup_{n\to+\infty} \sqrt[n]{|c_{-n}|}.$$

Then f_P converges on $\{|z-a| > r\}$.

Proposition. *With the notation above.*
1) If $0 \le r \le R \le +\infty$*, then the Laurent series converges on the annulus* $A(a;r,R) = \{r < |z-a| < R\}$*.*
2) If $r \ge R$ *then* f *diverges on any domain.*

Theorem. *1. Let* f *be a holo* f_n *on* $A(a;r,R)$ *with* $r < R$*. Then* f *has Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \forall r < |z-a| < R$$

where

$$c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \int_{|z-a|=\rho'}, \quad r < \rho < R.$$

For any $r < \rho' < R$*. And the expansion is unique.*

Proof. For $z \in A(a, r, R)$, pcik small $\epsilon > 0$ such that $z \in A(a; r+\epsilon, R-\epsilon)$. By Cauchy Integral Formula. We have that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|z-a|=R-\epsilon}\frac{f(\zeta)}{(\zeta - z)}d\zeta - \frac{1}{2\pi i}\int_{|z-a|=r+\epsilon}\frac{f(\zeta)}{(\zeta - z)}d\zeta \\ &= f_1 + f_2. \end{aligned}$$

For f_1 , write

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z-a)} \\ &= \frac{1}{\zeta - a}\frac{1}{1-\frac{z-a}{\zeta-a}} \\ &= \frac{1}{\zeta - a}\sum_{k=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^k, \end{aligned}$$

since $|z-a| < R-\epsilon = |\zeta - a|$ for all ,$\zeta \in \partial B(a, R-\epsilon)$. By Weirstrass Theorem we have that

$$\begin{aligned} \frac{1}{2\pi i}\int_{\partial B(a,R-\epsilon)}\frac{f(\zeta)}{\zeta - z}d\zeta &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|\zeta-a|=R-\epsilon}\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}(z-a)^n\frac{1}{2\pi i}\int_{|\zeta-a|=R-\epsilon}\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}c_n(z-a)^n, \end{aligned}$$

where $\int_{|\zeta-a|=R-\epsilon} = \int_{|\zeta-a|=\rho}$.

For f_2 , write

$$\begin{aligned} \frac{1}{z-\zeta} &= \frac{1}{(z-a) - (\zeta-a)} \\ &= \frac{1}{z-a}\frac{1}{1-\frac{\zeta-a}{z-a}} \\ &= \frac{1}{z-a}\sum_{k=0}^{\infty}\left(\frac{\zeta-a}{z-a}\right)^k, \end{aligned}$$

since $|z-a| > r+\epsilon = |\zeta - a|$ for all ,$\zeta \in \partial B(a, r+\epsilon)$. By Weirstrass Theorem we have that

$$\begin{aligned} \frac{1}{2\pi i}\int_{\partial B(a,R-\epsilon)}\frac{f(\zeta)}{\zeta - z}d\zeta &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|\zeta-a|=r+\epsilon}\frac{(\zeta-a)^n f(\zeta)}{(z-a)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}(z-a)^{-n}\frac{1}{2\pi i}\int_{|\zeta-a|=r+\epsilon}(\zeta-a)^{n-1}f(\zeta)d\zeta \\ &= \sum_{n=0}^{\infty}c_{-n}(z-a)^{-n}, \end{aligned}$$

where $\int_{|\zeta-a|=r+\epsilon} = \int_{|z-a|=\rho}$. (The trick for Lauren series is to try to write into geometric series.)

Uniqueness:

For uniqueness, assume $f(z)=\sum_{m=-\infty}^{\infty} d_m (z-a)^m$. Fix $k\in\mathbb{Z}$. Want $c_l = d_k$. Then

$$\frac{f(z)}{(z-a)^{k+1}} = \sum_{m=-\infty}^{\infty} d_m (z-a)^{m-k-1}.$$

Now

$$c_k = \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta-a)^{k+1}} d\zeta = \sum_{m=-\infty}^{\infty} d_m \int_{|z-a|=\rho} (\zeta-a)^{m-k-1} d\zeta.$$

Note using Polar coordinates with $\zeta = a+\rho e^{i\theta}$ and $d\zeta = \rho i e^{i\theta} d\theta$ we have that

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z-a|=\rho} (\zeta-a)^{m-k-1} d\zeta &= \frac{1}{2\pi}\rho^{m-k}\int_0^{2\pi} e^{i(m-k)\theta} d\theta \\ &= \rho^{m-k}\delta_{m,k} \\ &= \delta_{m,k}\end{aligned}$$

since

$$\int_0^{2\pi} e^{i(m-k)\theta} d\theta = \begin{cases} 0 & m\neq k \\ 2\pi & ,m=k \end{cases}.$$

□

Isolated Singularities:

- Let $a\in\mathbb{C}$ and f be a holo on $B(a;R)^{\star} = B(a;R)\setminus\{a\}$. So by Theorem 1,

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n (z-a)^n \\ &= f_H + f_P. \end{aligned}$$

Definition. 1) If $\lim_{z\to a} f(z)$ exists and finite, then a is a removable singularity of f.
2) If $\lim_{z\to a} f(z)=\infty$, then a is a pole of f.
3) If $\lim_{z\to a} f(z)$ does not exist, then a is an essential singularity of f.

Theorem. *2. Let f be holo on $B(a;R)^{\star}$. TFAE,*
i) a is a removarble singularity of f.
ii) There exists $0<\delta<R$ such that $\sup_{B(a;\delta)^{\star}} |f| < +\infty$.
iii) The principal part $f_P\equiv 0$. So $f\equiv f_H$.

Proof. Now $(i) \implies (ii)$ by definition.
Also $(ii) \implies (iii)$ For each $k\geq 1$, we have that for all $0<\rho<\delta$;

$$\begin{aligned} |c_{-k}| &= \left|\frac{1}{2\pi i}\int_{|z-a|=\rho} f(\zeta)(\zeta-a)^{k-1} d\zeta\right| \\ &\leq \sup_{0<|z-a|<\rho} |f| \frac{1}{2\pi}\int_{|z-a|=\rho} \left|(\zeta-a)^{k-1}\right| d\zeta \\ &= \sup_{0<|z-a|<\rho} |f| \frac{1}{2\pi}\int_{|z-a|=\rho} \rho^{k-1} d\zeta \\ &= \sup_{B(a;\rho)^{\star}} |f|\rho^k \to 0 \end{aligned}$$

as $\rho \to 0$.

$(iii) \implies (i)$ Now $f = f_P + f_H = f_H = \sum_{n=0}^{\infty} c_n (z-a)^n$ and $\lim_{z\to a} f(z) = c_0$ exists and is finite. □

Corollary. *(Riemann extension Theorem) Let f holomorphic on $B^\star(a;\delta)$. If*

$$|f(z)| \leq C \text{ on } B^\star(a;\delta), \qquad (\star)$$

then there exists a holomorphic F on $B(a;R)$ such that

$$F(z) = f(z) \;\; \forall z \in B^\star(a;R).$$

Remark. 1. Note that ($\star$) can be replaced by $|f(z)| < C \log \frac{1}{|z-a|}$, with $0 < |z-a| < \delta$ or

$$\lim_{z\to a} (z-a) f(z) = 0$$

Remark. 2. You may use Rieman extension Theorem so solve HW 4.1.

So image is bounded by 1 , so $D^\star$ it can be extended.

Theorem. *2. Let f be holomorphic on $B^\star(a;R)$: TFAE:*

i) a is a pole of f.

ii) There exists $0 < \delta < R$ such that

$$f(z) = \frac{h(z)}{(z-a)^m} \text{ on } B^\star(a;\delta) \text{ }, m \geq 1.$$

(iii) We have that

$$f_P = \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c_{-1}}{z-a},$$

where $c_{-m} \neq 0$. (a is a pole of order m, if $m = 1$ a is a simple pole.)

Proof. $(i) \implies (ii)$ Note that

$$\lim_{z\to a} \frac{1}{f(z)} = 0$$

We can assume $f(z) \neq 0$ on $B^\star(a;\delta)$ on a smaller punctured disk. Thus $\frac{1}{f}$ is holomorphic on $B^\star(a;\delta)$ and has removable singularity at a. Thus $\frac{1}{f}$ is holomorphic on whole disk$B(a;\delta)$. Now

$$\frac{1}{f} = h_1(z) (z-a)^m,$$

for $m \geq 1$. This implies that

$$f(z) = \frac{h_1^{-1}(z)}{(z-a)^m} = \frac{h(z)}{(z-a)^m}.$$

□

3/10/2015

- (Problem #1, HW3) A subtlety
  - f,g holo on $\mathbb{D}$. $f,g\in C^0\left(\overline{D}\right)$ then $|f|=|g|$ on $\partial\mathbb{D}$ which implies that $|f|\equiv|g|$ on $\overline{\mathbb{D}}$.
  - Natural Idea: Apply maximum principle. Let $h=\frac{f}{g}$. $h\neq 0$ on $\mathbb{D}$ so h is holo on $\mathbb{D}$. Assume $h\in C^0\left(\overline{D}\right)$. then
$$\max_{\overline{D}}|h|=\max_{\partial\mathbb{D}}|h|=1$$
so that $|f|\leq|g|$. Switch $h_1=\frac{g}{f}$ and get $|f|\geq|h|$
  - For example $f(z)=(z-1)^3$. Then for some g with $|g(z)|=|f(z)|=|z-1|^3$. If we let $h=\frac{g}{f}=\frac{g}{(z-1)^3}$. Can this guy still be continuously extended to the bunary of the disk?
  - We have to use analyticitiy. Say, $(|z|-1)^3=\left(|z|^2-1\right)^5$ on $\partial\mathbb{D}$. Then
$$\frac{\left(|z|^2-1\right)^3}{\left(|z|^2-1\right)^5}\notin C^0\left(\overline{\mathbb{D}}\right).$$
  - Try to fix this extention argument? (He'll give you extra credit.) His hint is that, actually you don't even need extention, just work on smaller disk. Walzona Weirstress. But he found a gap in his proof.
  - But this is a subtlety that he wants to mention.
- Last time:
- I. Let f be holo on $\{0<|z|<R\}$ Then $z=0$ is an isolated singularity of f. We say that
$$\begin{aligned} z=0 \text{ removable} &\iff \lim_{z\to 0} \text{ exists and finiter}\\ &\iff \sup_{B(0,\delta)}|f|<+\infty \text{ for some } 0<\delta<R\\ &\iff f \text{ holo on } B(0,R).\end{aligned}$$
- II. $z=0$ is a pole of order m.
$$\begin{aligned} &\iff \lim_{z\to 0}f(z)=\infty\\ &\iff f(z)=\frac{h(z)}{z^m} \text{ on } 0<|z|<\delta\\ &\iff f_P=\frac{c_{-m}}{(z-a)^m}+\cdots+\frac{c_{-1}}{z-a}\end{aligned}$$
on $B(0;\delta)$ where $c_{-m}\neq 0$.
- II. $z=0$ is an essential singularity.
$$\begin{aligned} &\iff \lim_{z\to 0}f(z) \text{ does not exist}\\ &\iff f_P=\sum_{n=1}^{\infty}\frac{c_{-n}}{z^n} \text{ on } B(0;\delta),\end{aligned}$$
whith infinitely many $c_{-n}\neq 0$.
- **Example:** The function $f(z)=e^{1/z}$ has essential dinsgularity at $z=0$.

– To see this we have

$$\lim_{z=x\to 0^+} e^{1/z} = \lim_{z\to 0^+} e^{1/x} = +\infty$$

and

$$\lim_{z=x\to 0^-} e^{1/z} = \lim_{z\to 0^-} e^{1/x} = 0.$$

Alternatively once can see that we can write

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{z^k k!}$$

with $\frac{1}{k!} \neq 0$.

Theorem. *1. (Casorati-Weierstrass) Let f be a holo on $B(a;R)^\star$ (punctured disk). Assume that a is an essential singularity of f. Then for any $0 < \delta < R$. We have*

$$\overline{f\left(B(a;R)^\star\right)} = \mathbb{C}$$

Proof. Suppose not. Then there exists $0 < \delta < R$ such that $\overline{f\left(B(a;R)^\star\right)} \subsetneq \mathbb{C}$. Pick $w_0 \in \mathbb{C}\backslash\overline{f\left(B(a;R)^\star\right)}$. Then

$$|f(z) - w_0| \geq \epsilon_0 > 0, \ \ \forall 0 < |z-a| < \delta.$$

Consider $g(z) = \frac{1}{f(z)-w_0}$. Thus

$$|g(z)| \leq \frac{1}{\epsilon_0} \quad \forall 0 < |z-a| < \delta.$$

Thus g has removable singularity at $z = a$, which implies that g is holomoprihc on $B(a;\delta)$ by Riemman extension theorem. Thus we can write

$$\frac{1}{f(z)-w_0} = \sum_{n=0}^{\infty} c_n (z-a)^n = (z-a)^m h(z)$$

on $B(a;\delta)$, for $m \geq 0$, (not infinitely many c_n's. ?) Thus

$$f(z) = \frac{h(z)^{-1}}{(z-a)^m} + w_0$$

has a pole or removable singularity at a. This is a contradiction. □

- **Remark:** Another way of obtaining a contradiction in the previous proof. Take

$$F(z) = \frac{f(z)-w_0}{z-a}, z \neq a.$$

Then take

$$\lim_{z\to a} |F(z)| = \lim_{z\to a} \left| \frac{f(z)-w_0}{z-a} \right| \geq \epsilon_0 \lim_{z\to a} \frac{1}{|z-a|} = \infty$$

which implies F has a pole. Which implies f has a pole or removable singularity at a.

- Let f be a holo on $\{0 < R < |z| < \infty\}$. Then we say ∞ is an **isolated singularity** on f. By changing variables, $w = \frac{1}{z}$. We have reduced to the isolated singularity at 0.

Proposition. *1.*
I. ∞ *is removable iff* $f(z) = \sum_{n=0}^{\infty} \frac{c_{-n}}{z^n}$ *on* $|z| > R$.
II. ∞ *is a pole of order* m *iff* $f(z) = \sum_{n=0}^{\infty} \frac{c_{-n}}{z^n} + c_1 z + \cdots + c_m z^m$ *with* $c_m \neq 0$.
III. ∞ *is essential iff*

$$f(z) = \sum_{n=0}^{\infty} \frac{c_{-n}}{z^n} + \sum_{n=1}^{\infty} c_n z^n$$

with inifinitely many $c_n \neq 0$ *for* $n \geq 1$.

Meromorphic Function:

- Recall f is an entire function if f is holomorphic on whole space $\mathbb{C}$.
- For entire functions, ∞ is an isolated singularity.
  - I. If ∞ is removable, then $f \equiv \text{cosnt}, c_0$. (by Prop 1 or Liouville Theorem).
  - II. If ∞ is a pole of order m, then

$$f(z) = c_0 + c_1 z + \cdots + c_m z^m, c_m \neq 0.$$

  - III. If ∞ is essential, then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

    with infintely many nonzero c_n.
  - Example are $e^z, \sin z, \cos z$, Bessel functions, etc.

Theorem. $Aut(\mathbb{C}) = \{\varphi(z) = az + b \mid a, b \in \mathbb{C}, a \neq 0\}$.

Proof. For all $f \in \text{Aut}(\mathbb{C})$. We write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \forall z \in \mathbb{C}.$$

There are finitely many nonzero c_n's. For otherwise, f has an essential singularity at ∞. By Weistrass Theorem, this would contradict $1-1$. A contradiction. Then

$$f(z) = c_0 + c_1 z + \cdots + c_m z^m.$$

Apply the open mapping theorem to

$$f(z) - c_0.$$

This implies that $f(z) = c_0 + c_1 z$ with $c_1 \neq 0$. □

- A function f is **meromorphic** (on $\mathbb{C}$) if f only has poles on $\mathbb{C}$.
- An enire function is meromorphic. .
- Besides, a rational function

$$R(z) = \frac{P_n(z)}{Q_n(z)}$$

  where $P_n(z) \propto Q_m(z)$ are reduced polynomials equations. $\frac{1}{z}$.

Proposition. *For rational function* R, ∞ *is either removable or a pole.*

Proof. Let $P_n(z) = a_n z^n + \cdots + a_0$ and $Q_n(z) = b_m z^m + \cdots + b_0$. Then

$$R(z) \sim \frac{a_n z^n}{b_m z^m} \to \begin{cases} \frac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \infty & \text{if } n > m \end{cases}$$

as $z \to \infty$. Thus $O(z^{n-m}) =$ that stuff , of order $n - m$. □

Theorem. *2. Let f be a meromorphic function. If ∞ is either removable or a pole, then f is rational.*

Proof. f is holo on $|z| > R$ □

Proposition. *1. Let $f(z) = f_H + f_{P,\infty}$ where $f_{P,\infty} = c_1 z + \cdots + c_m z^m$ on $|z| > R$.*

On $\{|z| \leq R\} = B\left(0; R\right)$ Then f has only finitely many poles. For otherwise, the limit point of poles is an essential singularity of f.

Denote the poles by $z_1, z_2, \ldots, z_p$. Then

$$f(z) = f_H + \frac{c_{-m_k}}{(z - z_k)^{m_k}} + \cdots + \frac{c_{-1}}{(z - z_k)}$$

near each $k = 1, \ldots, p$.

Claim. We have $f(z) = f_{P,\infty} + \sum_{k=1}^{p} f_{p,k} + \text{const}$ on $\mathbb{C}$. To see this, define

$$g = f - f_{p,\infty} - \sum_{k=1}^{p} f_{p,k}$$

for $\infty, z_1, \ldots, z_p$. Can show g has removable singularity at $\infty, z_1, \ldots, z_p$. This implies g is holomprihc, bounded on $\mathbb{C}$. This implies $g \equiv \text{const}$.

3/12/2015

- 1. Analytic Continuation
- 2. Residue Theorem
- 1. Let f be a holomorphic on domain U. If there exists a domain $G \supset U$ and a holomorphic F on G such that $F(z) = f(z)$ for all $z \in U$ then F is an **analytic continuation** of f.
- **Remark:** If F exists, then it is unique. (Recall if F_1, F_2 holo on G, then $F_1 = F_2$ on some $E \subset G$ where E contains a limit point in G then $F_1 \equiv F_2$ on G)
- **Example:** Power Series:
  - Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which has radius R of convergence. Which implies $f(z)$ is holo on $\{|z| < R\}$ by Abel's Theorem. For $\zeta_0 \in \partial B\,(0; R)$
  - (1) If $\exists B(\zeta_0, \delta)$ (smaller disk centered at the boundary) and g holo on $B\,(\zeta_0; \delta)$ such that $g(z) = f(z)$ for all $z \in B\,(\zeta_0, \delta) \cap B\,(0; R)$ then we say ζ_0 is a **regular point.**
  - (2) We say ζ_0 is a singular point og f if it is NOT regular.

Proposition. *i) The set of regular points of f is open in $\partial B\,(0; R)$.*
The set of singular points of f is closed in $\partial B\,(0; R)$.
ii) For $\zeta_0 \in \partial B\,(0; R)$, let z_0 be in $\overrightarrow{O\zeta_0}$ (ray from origin to ζ_0). Then $f(z)$ has Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n \quad a_n = \frac{f^{(0)}\,(z_0)}{n!}$$

Let ρ be the radius of convergence.
If $\rho > R - |z_0|$, then ζ_0 is a regular point.
If $\rho = R - |z_0|$, then ζ_0 is a singular point.

Proposition. *2. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius R of convergence. Then tehre exists one singular point on $\partial B\,(0, R)$.*

Proof. (Use Heine-Borel Theorem) Pick open cover of balls on the boundary circle. □

Example. 2. Show that every point on $\partial B(0, 1)$ is a singular point of

$$\sum_{n=0}^{\infty} z^{n!} \equiv f(z)$$

Proof. (1) f has radius of convergence $R = 1$. Let

$$a_k = \begin{cases} 1 & \text{if } k = n! \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\frac{1}{R} = \limsup_{k \to \infty} \sqrt[k]{|a_k|} = \limsup_{k_i \to \infty} |a_{k_i}|^{1/k_i} = \lim_{n \to \infty} 1^{1/n_i} = 1.$$

(2) It suffices to find a subset S of singular points of f such that S is dense in $\partial B(0; 1)$. Let

$$S = \left\{ e^{2\pi i \frac{p}{q}} \mid p \in \mathbb{Z}_{\geq 0}, q \in \mathbb{N}, p, q \text{ reduced} \right\}.$$

Then S is dense in $\partial B(0; 1)$. Fix an arbitrary $\zeta_1 \in S$. Want to show ζ_1 is singular. Write

$$f\,(r\zeta_1) = \sum_{n=0}^{q-1} r^{n!} e^{2\pi i \frac{pn!}{q}} + \sum_{n=q}^{\infty} r^{n!}.$$

Claim: $\varphi(r) \equiv \sum_{n=q}^{\infty} r^{n!} \to +\infty$ as $r \to 1^-$. If true, then we'd be done. For if so, then

$$\begin{aligned} |f(r\zeta_1)| &\geq \varphi(r) - \left| \sum_{n=0}^{q-1} r^{n!} e^{2\pi i \frac{pn!}{q}} \right| \\ &\geq \varphi(r) - q \\ &\to +\infty, \end{aligned}$$

as $r \to 1^-$. Supppose the claim is NOT true. Then

$$\lim_{r\to 1^-} \varphi(r) = M < +\infty.$$

(Note $\varphi(r)$ is increasing on $[0,1)$). Then $\forall N$, we have

$$\sum_{n=q}^{N} r^{n!} \leq M.$$

Letting $r \to 1^{-1}$, then $N - q \leq M$. But then this inequality would be true for any N, which is a contradiction. □

2. Residue Theorem

- Let f be holo on $B^{\star}(a;R)$. Define

$$\text{Res}(f;a) = \frac{1}{2\pi i} \int_{|z-a|=\rho} f(z)dz$$

where $0 < \rho < R$. Recall f has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(z)}{(z-a)^{n+1}} dz.$$

Proposition. *(1) If a is removable, then Res$(f;a) = 0$.*
(2) If a is a pole of order m, then

$$Res(f;a) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right).$$

In particular, if $m = 1$ then

$$Res(f;a) = \lim_{z\to a} \left[(z-a) f(z) \right].$$

Proof. (2) If a is a pole of order m, then using our third characterization of a pole of order m we have that

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c-1}{z-a} + f_H(z).$$

Then

$$(z-a)^m f(z) = c_{-m} + \cdots c_{-1}(z-a)^{m-1} + f_H(z)(z-a)^m$$

and so

$$\frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right) = c_{-1}(m-1)! + c f_H(z)(z-a).$$

so that

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)=c_{-1}(m-1)!$$

as needed. □

Theorem. *(Residue Theorem) Let γ be a rectifiable Jordan curve. Let U be a bounded domain enclosed by γ. Let f be a holomorphic function on $U\backslash\{z_1,\ldots,z_p\}$ and f is continuous on $\overline{U}\backslash\{z_1,\ldots,z_p\}$. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz=\sum_{j=1}^{p} Res\left(f;z_j\right).$$

Proof. Use Cauchy Integral Theorem to f on

$$\overline{U}\backslash\bigcup_{j=1}^{p}\overline{B\left(z_i,\delta_i\right)}$$

and fill in the details to

$$0=\int_{\gamma\cup\bigcup_{j=1}^{p}\partial B(z_i,\delta_i)} f(z)dz=\int_\gamma f(z)dz-\sum_{j=1}^{p}\text{Res}\left(f;z_j\right).$$

□

- **Remark:** We can define

$$\text{Res}\left(f;\infty\right)=-\int_{|z|=\rho} f(z)dz=-c_{-1}$$

for any holo f on $\{R<|z|<+\infty\}$.

Corollary. *Let f be a meromorphic function with poles $\{z_1,\ldots,z_p\}$. Then*

$$\sum_{j=1}^{p} Res\left(f;z_j\right)+Res\left(f;\infty\right)=0.$$

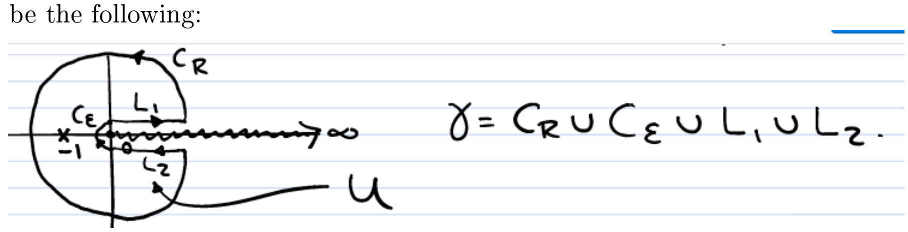
- **Example 2:** Compute

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx \text{ for } 0<p<1.$$

  - Consider

$$f(z)=\frac{z^{p-1}}{1+z}.$$

  Note that $z^{p-1}=e^{(p-1)\log z}$. We know that log has branch cuts at 0 and ∞. So we need to find a branch cut. (If we take the standard branch cut of $\log z$ then we get nothing!)
  - Let γ be the following:

* 

  - : Let R big number bigger than 1, and $\epsilon<1$ is small number. $\gamma=C_R\cup C_\epsilon\cup L_{1,2}$

– Here we take the branch cut to be $[0, +\infty)$ of $\log z = \log r + i\theta$ where $0 < \theta < 2\pi$ holo on U.
– So $f(z) = \frac{z^{p-1}}{z+1}$ is holo on $U \setminus \{-1\}$. Then

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \text{Res}\,(f;-1)\,.$$

Then

$$\frac{1}{2\pi i}\int_{C_R} f + \frac{1}{2\pi i}\int_{C_\epsilon} f + \frac{1}{2\pi i}\int_{L_1} f + \frac{1}{2\pi i}\int_{L_2} f = \text{Res}\,(f;-1)\,.$$

Note $\text{Res}\,(f;-1)$ " $= (-1)^{p-1}$ " $= \left(e^{i\pi}\right)^{p-1} = e^{i\pi(p-1)}$.

– Claim1: $\frac{1}{2\pi i}\int_{C_R} f \to 0$ and $\frac{1}{2\pi i}\int_{C_\epsilon} f \to 0$ as $R \to +\infty$ and $\epsilon \to 0^+$. where $C_R = \left\{Re^{i\theta} : \delta \leq \theta \leq 2\pi - \delta\right\}$ and $C_\epsilon = \left\{\epsilon e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$, and $L_1 = \{x + Ei \mid 0\text{or } \delta_1 \leq x \leq R, \epsilon > E > 0\}$ and $L_2 = \{x - Ei \mid 0 \leq x \leq R, 0 < E < \epsilon\}$.
– Then

$$\int_{L_1} f(z)dz = \int_{\delta_1}^{R} \frac{(x+\epsilon i)^{p-1}}{1+x+\epsilon i}dx$$

and so

$$\begin{aligned}
(x+\epsilon i)^{p-1} &\to x^{p-1} \\
x + \epsilon i &= \sqrt{x^2+\epsilon^2}e^{\left(\tan^{-1}\frac{\epsilon}{x}\right)i} \\
(x+\epsilon i)^{p-1} &= \sqrt{x^2+\epsilon^2}e^{\left(\tan^{-1}\frac{\epsilon}{x}\right)i(p-1)} \to x^{p-1}.
\end{aligned}$$

– Now

$$\int_{L_2} f(z)dz = \int_{R}^{0} \frac{(z-\epsilon i)^{p-1}}{1+x-\epsilon i}dx$$

where $z = x - \epsilon i$ then

$$\begin{aligned}
(x-\epsilon i) &= \sqrt{x^2+\epsilon^2}e^{\left(2\pi-\tan^{-1}\frac{\epsilon}{x}\right)i} \\
(x-\epsilon i)^{p-1} &= \sqrt{x^2+\epsilon^2}e^{\left(2\pi-\tan^{-1}\frac{\epsilon}{x}\right)i(p-1)} \to x^{p-1}e^{2\pi(p-1)i}.
\end{aligned}$$

– **Claim 2:** We have that

$$\begin{aligned}
\int_{L_1} f(z)dz &\to \int_0^{+\infty} \frac{x^{p-1}}{x+1}dx \\
\int_{L_2} f(z)dz &\to \int_{+\infty}^{0} \frac{x^{p-1}}{x+1}dx \cdot e^{2\pi(p-1)i}.
\end{aligned}$$

  * By Residue Theorem we have

$$\frac{1}{2\pi i}\int_0^{+\infty} \frac{x^{p-1}}{x+1}dx\left[1 - e^{2\pi(p-1)i}\right] = e^{\pi(p-1)i}$$

  and so

$$\begin{aligned}
\int_0^{+\infty} \frac{x^{p-1}}{x+1}dx &= \frac{2\pi i}{1-e^{2\pi(p-1)i}}e^{\pi(p-1)i} \\
&= \frac{2\pi i}{e^{-\pi(p-1)i} - e^{2\pi(p-1)i}} \\
&= \frac{\pi}{\sin p\pi}.
\end{aligned}$$

∗ The Claim follows from Jordan's Lemma or direct calculation.

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_R} f(z)dz\right| &\le \frac{1}{2\pi}\int_{\delta}^{2\pi\delta}|f(z)|\,dz,\text{ let } z = Re^{i\theta}\\ &\le \frac{1}{2\pi}\int_{\delta}^{2\pi}\frac{R^{p-1}R}{R-1}d\theta \sim cR^{p-1}\to 0\end{aligned}$$

as $R \to +\infty$.

Midterm 2 will be problems similar to the homework problems.

3/26/2015

Riemann Mapping Theorem

- One of two most important theorems in complex analysin. The other one is Cauchy Integral formula.

Theorem. *(A) Let $\Omega \subsetneq \mathbb{C}$ be a simply-connected domain in $\mathbb{C}$. Then Ω is biholomorphic to $\mathbb{D} = \{|z| < 1\}$. (That is, there exists holomorphic $f : \Omega \to \mathbb{D}$ such that f is onto and one-to-one)*

Remark. (1) If $\Omega = \mathbb{C}$, then Not true (Liuoville Theorem)

(2) Theorem implies Any two simply connected domains $\Omega_i \subsetneq \mathbb{C}$ are biholo to each other. This implies Ω_1 and Ω_2 are (diffeomorphic/homeomorphic) to each other. Here we can allow the domains to be equal to $\mathbb{C}$.

There exists differoemorphic map $\mathbb{C} \to \mathbb{D}$. Take $(x, y) \mapsto \left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}}\right)$.

($\star$)(3) Uniformization Theorem(Math 5121) Any Riemann surface has a universal cover, which is biholo to one of the following

i) $\mathbb{D}$, ii) $\mathbb{C}$ and iii) $\hat{\mathbb{C}}$.

- Our proof follows ideas of Schwarz, Poincare, etc.
- First a Lemma.

Lemma. *1. Let Ω be a simply-connected domain and f holo on Ω, $f \neq 0$ on Ω. Then we can take an analytic branch $g(z)$ of $\log f$ on Ω. That is, $g(z)$ holo on Ω satisfying*

$$e^{g(z)} = f(z), \forall z \in \Omega.$$

Proof. Note $\frac{f'}{f}$ is holomorphic on Ω. By Homework 2.5, there exists a primitive function g holomorphic on Ω such that $g' = \frac{f'}{f}$ on Ω.(We used simply connectedness here). Let $F(z) = f(z)e^{-g(z)}$ for $z \in \Omega$. Then

$$F'(z) = f'(z)e^{-g(z)} - e^{-g(z)}g'(z)f(z) \equiv 0$$

which implies $F(z) \equiv c$. This implies $e^{g(z)} = cf(z)$ so that $g(z)+\log c$ is branch of f. (Here $\log c = \log|c|+i\theta_0$ with $-\pi < \theta_0 < \pi$) □

Corollary. *2. Let Ω be simply connected and $f|_{\Omega} \neq 0$, holo. Then there exists a branch g of $\sqrt{g(z)}$ on Ω, $g|_{\Omega} \neq 0$.*

Proof. We have $\sqrt{f(z)} \overset{\text{def}}{=} e^{\frac{1}{z}\log f(z)}$. □

- We also need the following

Theorem. *(Montel) Let $\mathcal{F}$ be a family of holomorphic functions on Ω. If there exists a constant $M > 0$ such that $|f(z)| < M$, for all $f \in \mathcal{F}$ for all $z \in \Omega$, then $\mathcal{F}$ is normal, i.e. every sequence $\{f_k\}$ has a subsequence $\{f_{k_l}\}$ which convegres uniformly to a function f on every compact set of Ω.*

- Normal does not guarantee that the limiting funciton is in $\mathcal{F}$.

Theorem. *(B) Let $\Omega \subsetneq \mathbb{C}$ be a simply conneted domain. Fix $a \in \Omega$. Then there exists a holomorphic $f : \Omega \to \mathbb{D}$ such that*
1) $f(a) = 0$, and $f'(a) > 0$
2) f is one-to-one
3) f is onto.

Proof. **Uniqueness:**
The uniquenes is trivial. Suppose $f, g : \Omega \to \mathbb{D}$ satifies the properties. Then consider

$$\mathbb{D} \overset{f^{-1}}{\to} \Omega \overset{g}{\to} \mathbb{D}$$

which implies $g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ is biholomorphic, and $g \circ f^{-1}(0) = 0$. From a lemma about the automorphic group of $\mathbb{D}$ we have that $g \circ f^{-1}$ must be a rotation. That is, $g \circ f^{-1} = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$. Thus

$$g(z) = e^{i\alpha} f(z)$$

so that

$$g'(z) = e^{i\alpha} f'(z)$$

but

$$0 < g'(z) = e^{i\alpha} f'(z)$$

which implies $\alpha = 0$. Thus $g = f$ as needed.
Existence:
Let $\mathcal{F} = \{f : \Omega \to \mathbb{D} \text{ holo } f(a) = 0, f'(a) > 0, \ f \text{ is one-to-one } \}$.
Step1: $\mathcal{F} \neq \emptyset$
Step2: There exists $g \in \mathcal{F}$ such that $g'(a) = \sup \{f'(a) : f \in \mathcal{F}\}$. (This is the hardest)
Step3: g is onto.
Proof of Step1:
Since $\Omega \subsetneq \mathbb{C}$, there exists a $b \in \mathbb{C} \backslash \Omega$. By Corollary 2, there exists branch g of $\sqrt{z-b}$, $z \in \mathbb{C}$ with $g \mid_{\Omega} \neq 0$.
Claim(1) g is one-to-one. Claim (2) For all $z \in \Omega$,$-g(z) \notin \Omega$.
In fact, $[g(z_1) - g(z_2)] [g(z_1) + g(z_2)] = z_1 - z_2$ for all $z_1, z_2 \in \Omega$. This implies (1) and (2). Let

$$f(z) = e^{i\alpha} \beta \frac{g(z) - g(a)}{g(z) + g(a)}, \forall z \in \Omega.$$

Here α and β are real constant TBA. Note f is holo. $f(a) = 0$ and f is 1-1. Remains to check $f'(a) > 0$ and $f(\Omega) \subset \mathbb{D}$. Now

$$\begin{aligned} f'(a) &= \left[e^{i\alpha} \beta \frac{g'(z) [g(z) + g(a)] - (g(z) - g(a)) g'(z)}{[g(z) + g(a)]^2} \right]_{z=a} \\ &= e^{i\alpha} \beta \frac{g'(a)}{2g(a)} > 0 \end{aligned}$$

if we set

$$\alpha := e^{i\alpha} \frac{g'(a)}{g(a)} = \left| \frac{g'(a)}{g(a)} \right| > 0.$$

Now

$$|f(z)| = |\beta| \left[1 + \left|\frac{2g(a)}{g(z)+g(a)}\right|\right].$$

We want $[g(z)+g(a)] >?$. By open mapping Theorem, there exists $\overline{B(g(a),\delta)} \subset f(\Omega)$ for some small $\delta > 0$. Then

$$|g(z)+g(a)| \geq \delta, \ \ \forall z \in \Omega.$$

This implies $|f(z)| \leq \beta \left(1 + \frac{2|g(a)|}{\delta}\right) < 1$ for small $\beta > 0$. This completes Step1.

Proof of Step2:

Denote $B \equiv \sup\{f'(a) : f \in \mathcal{F}\}$. Since its a supremum then there exists a sequence $\{f_n\} \subset \mathcal{F}$ such that $f_n'(a) \to B$. Since $|f_n(z)| \leq 1$, for all n and for all $z \in \Omega$. By Montoel's Theorem, $f_{n_k}(z) \rightrightarrows_{\text{cpt}} g(z)$. Then g is holo, and $g'(a) = \lim f_n'(a) = B > 0$ which implies that $0 < B < +\infty$. Thus

$$|g(z)| \leq 1, \forall z \in \Omega.$$

Remains to check g is 1-1, which follows from Hurwitz Theorem.

Proof of Step3:

Suppose NOT. That means there exists a $w_0 \in \mathbb{D}\backslash g(\Omega)$. Let

$$F(z) = \sqrt{\frac{g(z)-w_0}{1-\overline{w_0}g(z)}}$$

be a branch. We can do this because since w_0 is not in the image then $1-\overline{w_0}g(z)$ is never vanishing. The inside is like a Mobius transformation. Now you cna check that

$$|F(z)| \leq \left|\frac{g(z)-w_0}{1-\overline{w_0}g(z)}\right| \leq 1,$$

using the estimate $|\sqrt{z}| = |z|^{\frac{1}{2}} \leq |z|$ when $|z| < 1$. Thus F is one-to-one. Want to modify F to be $\tilde{F} \in \mathcal{F}$. Let

$$\tilde{F}(z) = e^{i\gamma}\frac{F(z)-F(a)}{F(z)+F(a)},$$

where $\gamma \in \mathbb{R}$ TBA(he means, to be determined). Now $\tilde{F}(a) = 0$. Furthermore since $\sqrt{z}' = \frac{1}{2\sqrt{z}}$ then

$$\begin{aligned} \tilde{F}'(a) &= \left[e^{i\gamma}\frac{F'(z)\left[F(z)+F(a)\right]-\left[F(z)-F(a)\right]F'(z)}{\left[F(z)+F(a)\right]^2}\right]_{z=a} \\ &= e^{i\alpha}\frac{F'(a)}{2F(a)}. \end{aligned}$$

Computing we get that $F(a) = \sqrt{-w_0}$ and

$$F'(a) = \frac{1}{2}\left(\sqrt{-w_0}\right)^{-1}\frac{\left(1-|w_0|^2\right)g'(a)}{\left[1-\overline{w_0}g(a)\right]^2}.$$

Pluggin these back into $\tilde{F}'(a)$ we get

$$\tilde{F}'(a) = \frac{1+|w_0|}{2\sqrt{|w_0|}}g'(a) > g'(a)$$

which is a contradiction. (Since $1+a^2 > 2a, \forall a$).

Remark: Idea: Get maps

$$\mathbb{D} \overset{\tilde{F}^{-1}}{\to} \Omega \overset{g}{\to} \mathbb{D} \backslash \{w_0\}$$

by $0 \mapsto a \mapsto 0$ so that $g \circ \tilde{F}^{-1} : \mathbb{D} \to \mathbb{D}$ can be applied Schwarzz lemma so that

$$|g'(a)| \left| \tilde{F}'(a) \right|^{-1} < 1,$$

as needed. □

4/31/2015

- HW4-#1 $f : \mathbb{D}^\star \to \mathbb{D}^\star$ we show $f(z) = e^{\alpha i} z$ for $\alpha \in \mathbb{R}$ fixed.
  - Key is to show $f(0) = 0$.
  - Quizk Way: Riemann extension $f(0)$ is defined. f holo on $\mathbb{D}$. Open mapping theorem. Now $B(f(0), \delta) \subset f(\mathbb{D})$. Pick small neighborhood on boundary, inside and intersection 0 and get a contradiction.
  - Another way: Riemann extension to $f^{-1}(0)$.
  - He said make sure you can solve it!!!!!!!!!!!!!!!!!! Maybe this is a hint that it will be on the PRELIM!!!!!!!!!!!!!!!!!!!!!!!!
- HW4-#3
  - Cannot define $\int_{z_0}^{z} g dz$ on $0 < |z| < 1$. You want to make sure that every curbe has the same valued. Like is $\int_{\gamma_1} g dz \neq \int_{\gamma_2} g dz$?! So its not well defined. The mean value theorem for analuytic functions does not hold.
  - Method 1: Since its isolated singularity we can write

$$\begin{aligned} f_P(z) &= \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots \\ f_P'(z) &= \frac{-c_{-1}}{z^2} + \frac{\tilde{c_{-2}}}{z^3} + \cdots \end{aligned}$$

then

$$\begin{aligned} f &= f_P + f_H \\ f' &= f_P' + f_H', \end{aligned}$$

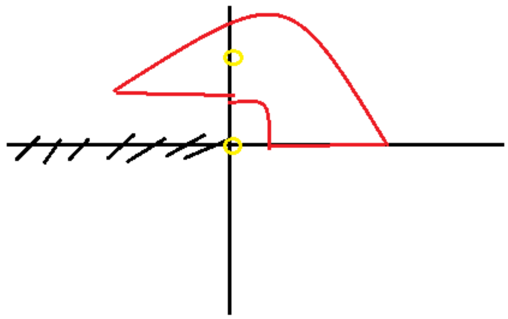
okay.
  - Method2: Take $f'(z) = \frac{1}{z}$. So

$$0 = \int_{\partial B(0,\delta)} \frac{1}{z} dz = 2\pi i \neq 0.$$

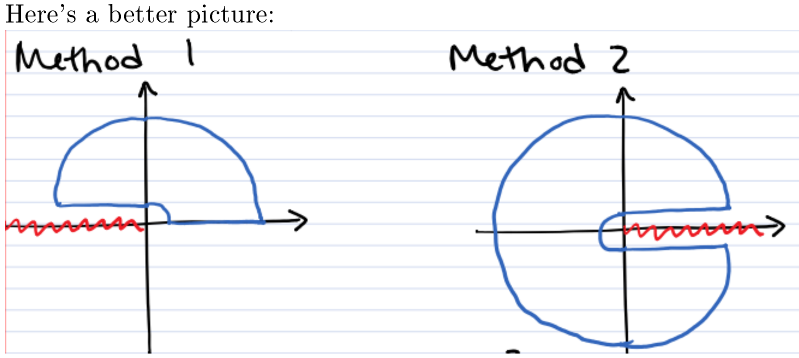
You want to integrate on a specific curve.
- HW4 - #5-
  - Use Schwarz lemma.
  - Get

$$\sum |f(z^n)| \leq \sum |z^n| \leq \sum |z|^n = \frac{1}{1 - |z|} < \infty$$

as needed.
  - Now $\left\{ \frac{k}{2^n} \right\}$ is dense in $[0, 1)$.
  - Can also check that $f(z) = z + f\left(z^2\right)$. Rudin's theorem. Hadaman's theorem.
- Test2- #1
  - Now for $|z| \geq 1$ get $|f(z)| \leq e^{1/|z|} \leq e$ and for $|z| \leq 1$ get $|f(z)| \leq \max_{|z|=1} |f(z)| \leq C$ so $f \equiv \text{const} = 0$
- Test2 - #2
  - For $\int_0^\infty \frac{\log x}{x^2 + a^2} dx$.
  - Method 1: Choose branch $-\pi < \theta < \pi$.



*
* Here's a better picture:



*

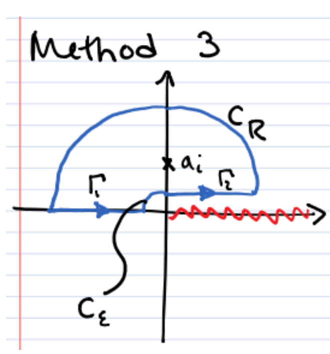
- Method 2: Choose Pacman. Then let branch be defined on $0 < \theta < 2\pi$.
- Get

$$\int_0^R \frac{\log z}{z^2+a^2}dz + \int_R^0 \frac{\log z + 2\pi i}{z^2+a^2}dz = \int_0^R \frac{2\pi i}{z^2+a^2}dz$$

  which is not usefull.
- Thus Can use $\frac{(\log z)^2}{z^2+a^2}$. Then

$$\int_0^R \frac{(\log z)^2}{z^2+a^2} - \int_0^R \frac{(\log x + 2\pi i)^2}{x^2+a^2} = -\int_0^R 2\pi i \frac{\log x}{x^2+a^2}dx.$$

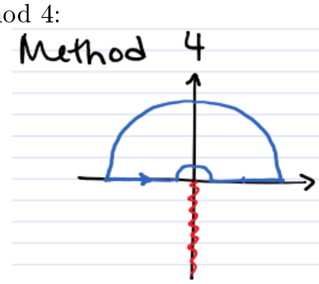
- Method 3.



* Picture:
* Use branch $0 < \theta < 2\pi$.

* Get $\operatorname{Res}\left(\frac{\log z}{z^2+a^2}; ai\right) = \frac{\log a + \frac{\pi}{2} i}{2ai}$. Then $\Gamma_1 = \{-x; 0 < x < R\}$. Then
$$\int_{\Gamma_1} = \int_0^R \frac{\log z + \pi i}{x^2+a^2} dx$$
and $\Gamma_2 = \{x + \epsilon i \mid 0 < x < R\}$ so that
$$\int_{\Gamma_2} = \int_0^R \frac{\log x}{x^2+a^2} dx.$$
Then
$$\left|\int_{C_\epsilon}\right| \leq K\epsilon \log \epsilon \to 0 \text{ as } \epsilon \to 0$$
$$\left|\int_{C_R}\right| \leq K_1 \frac{1}{R} \to 0 \text{ as } R \to +\infty.$$
Then
$$2\int_0^{+\infty} \frac{\log x}{x^2+a^2} dx + \pi i \int_0^{+\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}\left(\log a + \frac{\pi}{2} i\right).$$

– Method 4:

* 

- Similar problem: $\int_0^\infty \frac{\sqrt{x}\log x}{x^2+a^2} dx$.

Theorem. *(Riemann Mapping Theorem) For every $\Omega \subsetneq \mathbb{C}$ that is simply connected. Fix $z_0 \in \Omega$. There exists a unique holomorphic function $f : \Omega \to \mathbb{D}$ such that*
(1) $f(z_0) = 0$ and $f'(z_0) > 0$.
(2) $f : \Omega \to \mathbb{D}$ is 1-1.
(3) $f : \Omega \to \mathbb{D}$ is onto.

Proof. From last time: We have that
$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \text{ holo} \mid f \text{ satisfies (1) and (2)}\}.$$
Lemma(1): $\mathcal{F} \neq \emptyset$.
Lemma(2): There exists $g \in \mathcal{F}$ such that $g'(a) = \sup_{f\in\mathcal{F}} f'(a)$.
Lemma(3): g is onto.
For Lemma(2) recall that we need
Theorem (Montel): Let $\Omega \subset \mathbb{C}$ be a domain $\mathcal{F}$ a family of holo on Ω. Now f there exists a constant $M > 0$ such that
$$\sup_{f\in\mathcal{F}} \sup_{z\in\Omega} |f(z)| \leq M$$
then $\mathcal{F}$ is normal.
——————————-

Also need:
Lemma (Ascoli-Arzela): Let f_n be a sequence of functions on a **compact set** $K \subset \mathbb{R}^N$ such that:
(1) f_n is equibounded i.e. there exists $c > 0$ such that

$$\sup_n \sup_{z \in K} |f_n(z)| \leq C.$$

(2) f_n is equicontinuous, i.e. for every $\epsilon > 0$ ther eexists $\delta > 0$ such that

$$\sup_n |f_n(x) - f_n(y)| < \epsilon,$$

for every $|x - y| < \delta$. Then there existsf_{n_k} of f_n such that $f_n \rightrightarrows f$ on K.
——————-

Remark: For Lemma AA, one need K to be compact. For exan we can finf a sequneve of smooth functions

$$f_n(x) = \begin{cases} 1, & x \geq n \\ 0 & x < n-1 \end{cases}.$$

Then clearly $|f_n| \leq 1$ and $|f_n'| \leq C$.

Now (AA) implies Theorem (Motel)

Proof: Let $\{f_n\} \subset \mathcal{F}$. Then $\{f_n\}$ is equibounded. Need to show f_n is equicontinuous. Write

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial B(\delta)} \frac{f_n(\zeta)}{\zeta - z} d\zeta$$

Then for all $z, w \in B(\delta)$ we get

$$\begin{aligned} |f_n(z) - f_n(w)| &= \left| \frac{1}{2\pi i} \int_{\partial B(\delta)} f_n(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{\partial B(\delta)} |f_n(\zeta)| \frac{|z - w|}{(\zeta - z)(\zeta - w)} d\zeta \\ &\leq \frac{M}{2\pi} |z - w| \frac{1}{\delta^2/4} \\ &\leq C |z - w| \\ &< \epsilon. \end{aligned}$$

By chososing small enough ball.

Remark: Method2: Can first show that $|f'(z)| \leq C$. Then

$$\begin{aligned} |f_n(z) - f_n(w)| &= \left| \int_\gamma f_n'(\gamma(t)) \gamma'(t) dt \right| \\ &\leq C_1 |z - w| < \epsilon \end{aligned}$$

by pick γ to be a line segment from z to w, and then realizing that its length is $|z - w|$.

One cannot argue as follows: $|f_n(z) - f_n(w)| \neq |f_n'(\eta)(z - w)| \leq |f'(\eta)| |z - w| \leq C |z - w|$.

Example: Take $f(z) = e^z$ then $e^{2\pi i} = e^{0i} = 1$. Then $e^{2\pi i} - e^{0i} = e^\eta (2\pi i - 0)$. Contradiction

Remark2: Montel's theorem can be relaxed. It turned out you don't have to find a global bound. But you can bound the family on a smaller compact neighbrhood. Just like the Homework Problem:

This revisied Montell's theorem will have an improved version, down at the bottom. □

Relevant Theorems from above:

Theorem. *(2)* ***Montel's Theorem*** *(Improved version) Let $\Omega \subset \mathbb{C}$. Let $\mathcal{F}$ be holomorphic on Ω. Suppose for every compact set $K \subset \Omega$, there exists constant $M_K > 0$ such that*

$$\sup_{f\in\mathcal{F}} \sup_{z\in K} |f(z)| \leq M_K,$$

then $\mathcal{F}$ is ***normal.***

- Like in the Homeowork, you can bound $\frac{1}{1-r}$ on $|z| < \delta < 1$ but not when we iunclude 1.

Lemma. ***(Ascoli-Arzela) or (AA):*** *Let f_n be a sequence of functions on a* ***compact set*** *$K \subset \mathbb{R}^N$ such that:*

(1) f_n is equibounded i.e. there exists $c > 0$ such that

$$\sup_n \sup_{z\in K} |f_n(z)| \leq C.$$

(2) f_n is equicontinuous, i.e. for every $\epsilon > 0$ ther eexists $\delta > 0$ such that

$$\sup_n |f_n(x) - f_n(y)| < \epsilon,$$

for every $|x - y| < \delta$. Then there existsf_{n_k} of f_n such that $f_n \rightrightarrows f$ on K.

Proof. There exists $E \subset K$ which is countable and dense in K. $(E - K \cap \mathbb{Q}^N)$, with $E = \{x_1, x_2, \dots\}$. For x_1, we have that $\{f_n(x_1)\}_{n=1}^{\infty}$ has a convergent subsequence. Say $f_{n_k} := f_{n_{k_1}}, f_{n_{k_2}}, \dots$. For x_2, look at $\{f_{n_k}(x_2)\}$ and find its convergent subsequence. Say $f_{n_{k_1}}, f_{n_{k_2}}, \dots$. Then this the usual diagonilization argument. And look at the diagonal subsequence. Pick the diagonal $\{f_{nn}\}$. Then f_{nn} converges on E. This is because

$$|f_{nn}(x) - f_{mm}(y)| \leq |f_{nn}(x) - f_{nn}(x')| + |f_{mm}(y) - f_{mm}(y')|$$

And then use equicontinuity and fact that $E \subset K$. □

4/7/2015

Theorem. *(Riemann Mapping Theorem) Let $\Omega \subsetneq \mathbb{C}$ simply-connected domain, there exists a biholomorphic $f : \Omega \to \mathbb{D}$.*

- Can we extend f continuously to $\partial\Omega \to \partial\mathbb{D}$ such that $f : \overline{\Omega} \to \overline{\mathbb{D}}$.
- A neccessary condition is Ω has to be bounded.
  - For, $\overline{\Omega} = f^{-1}\left(\overline{\mathbb{D}}\right)$ compact. Assume Ω is bounded. The answer is Yes, if $\partial\Omega$ satisfies certain condition.
- Example1: $\partial\Omega$ is very "nice". (i.e., $\partial\Omega$ is smooth) Let $\Omega = \mathbb{D}$ and $\partial\Omega = \partial\mathbb{D} = S^1$.
  - Every function $f : \mathbb{D} \to \mathbb{D}$ biholomrphic is of the form
  $$f(z) = e^{i\alpha}\frac{z-b}{1-\bar{b}z}$$
  for $\alpha \in \mathbb{R}, b \in \mathbb{D}$ and $|b| < 1$.
  - For $|z| = 1$, we have that
  $$\begin{aligned} |f(z)| &= \left|\frac{z-1}{1-\bar{b}z}\right| \\ &= \frac{|\bar{z}|\,|z-b|}{\left|1-\bar{b}z\right|} \\ &= \frac{|1-b\bar{z}|}{\left|1-\bar{b}z\right|} = 1. \end{aligned}$$
  In fact , $f \in \operatorname{Aut}(\partial\mathbb{D})$.
  - f is holo since $\frac{\partial}{\partial\bar{z}}f = 0$. So $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ biholo.
- Example2: Let $\Omega = \mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}$. Want $\mathbb{D}_+ \to \mathbb{D}$. Then $L_1 = \frac{z+1}{z-1}$ and $R = e^{i\pi}z$ and $f = z^2$ and $g = \frac{z+i}{z-i}$. Then L_1 bring top hald disk to third quadrant, then $e^{i\pi}$ rbing to first qudrant then bring to the top half plane. Get
  $$\begin{aligned} F &= g \circ f \circ R \circ L_1 \\ &= \frac{\left(\frac{z+1}{z-1}\right)^2 + i}{\left(\frac{z+1}{z-1}\right)^2 - i} \\ &= \frac{(z+1)^2 + i\,(z-1)^2}{(z+1)^2 + i\,(z-u)^2}. \end{aligned}$$
  F can extend continuously to $\overline{\mathbb{D}_+}$. But $F \notin C^1\left(\overline{\mathbb{D}_+}\right)$, since $F'(\pm 1) = 0$. $\left(F^{-1} = \sqrt{z+1}\right)$.
- Example3: Let
  $$\Omega = \mathbb{D}\backslash \left\{x + yi \mid y = 0, 0 \le x < 1\right\}.$$
  Find $G : \Omega \to \mathbb{D}$ biholo. Then $G = F \circ \sqrt{z}$. G cannot extend continuous to $\Gamma = \partial\Omega$. Say, pick $\frac{1}{4} \in \Gamma$.
  - Bring $\Omega \to \mathbb{D}_+$ (top have disk)$\to \mathbb{D}$.
  - Then $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ for $0 < \theta < 2\pi$.
  - Then $z_n = \frac{1}{4}e^{i/n} \to \frac{1}{4}$ and $w_n = \frac{1}{4}e^{i\left(2\pi - \frac{1}{n}\right)} \to \frac{1}{4}$.
  - But $\sqrt{z_n} \to \frac{1}{2}$ and $\sqrt{w_n} \to -\frac{1}{2}$. With $F\left(\frac{1}{2}\right) \ne F\left(-\frac{1}{2}\right)$. And $G(z_n) \to F\left(\frac{1}{2}\right) \ne F\left(-\frac{1}{2}\right) \leftarrow G(w_n)$.

Theorem. *1. Let Ω be a bounded simply-connected domain in $\mathbb{D}$. If $\partial\Omega$ is a single/simple? Jordan curve, then any biholomorphic $f:\Omega\to\mathbb{D}$ can extend to a homeomorphism $\overline{\Omega}$ onto $\overline{\mathbb{D}}$.*

Proof. **Step1:** f can be defined on $\partial\Omega$.

Step2: $f\in C^0\left(\overline{\Omega}\right)$

Step3: f is 1-1.

Then we are done. For $\mathbb{D}\subset f\left(\overline{\Omega}\right)$ is compact and $f\left(\overline{\Omega}\right)\subset\overline{\mathbb{D}}$. This implies that $f\left(\overline{\Omega}\right)=\overline{\mathbb{D}}$. Then f^{-1} is continuous. Any continuous function form a compact space to a Hausdorff space implies that f^{-1} is continuous.

Lemma1: For every $\zeta\in\partial\Omega$, the limit

$$\lim_{z\to\zeta,\,z\in\Omega} f(z) \text{ exists.}$$

Consequently, we can define

$$f(\zeta)=\lim_{z\to\zeta,\,\zeta\in\Omega} f(z),$$

Proof: Oberseve the fact $\forall\{z_n\}\subset\Omega$ with $z_n\to\zeta\in\partial\Omega$. If $f(z_n)\to f(\zeta)\in\overline{\mathbb{D}}$, then $|f(\zeta)|=1$.

Proof of fact: Suppose $|f(\zeta)|<1$. Let $g=f^{-1}$. Then $\zeta=g(f(\zeta))=\lim_{n\to\infty} g(f(z_n))=\lim_{n\to\infty} z_n=\zeta$. Since $g:\mathbb{D}\to\Omega$, and $\zeta=g(f(\zeta))\in\Omega$. This contradicts $\zeta\in\partial\Omega$. Suppose $\lim_{z\to\zeta,z\in\Omega} f(z)$ does NOT exist. Then there exists $\{z_n\zeta\};\{w_n\}\subset\Omega$ such that $\lim_{n\to\infty} f(z_n)=a$ and $\lim_{n\to\infty} f(w_n)=b$ where $a,b\in\mathbb{C}$ and $a\neq b$. Then $(f(\Omega)\subset\mathbb{D})$ Furthermore, $|a|=|b|=1$, by previous Fact.

Proposition: For all $a,b\in\partial\mathbb{D}$, there exists a Mobius transform φ such that $\varphi(a)=1$ and $\varphi(b)=-1$.

Proof: Have that with

$$\varphi=\frac{z-\alpha}{1-\bar{\alpha}z}$$

$\varphi(a)=1$ and $\varphi(b)=-1$ then implies

$$\alpha=\frac{z+\bar{b}-\bar{a}}{\bar{b}+\bar{a}}.$$

QED.

Now the proposition implies that there exists φ such that $F=\varphi\circ f:\Omega\to\mathbb{D}$. Then

$$\lim_{n\to\infty} F(z_n)=e^{i\pi/4} \text{ and } \lim_{n\to\infty} F(w_n)=e^{i5\pi/4}.$$

Now let $\zeta_0=F^{-1}(0)$ and $d\equiv|\zeta_0-\zeta|>0$. Now for all $0<\delta<\frac{d}{2}$. Then for $z_n,w_n\in B(\zeta;\delta)\cap\Omega$ for all $n\geq N$. Then let γ be a continuous curve in $B(\zeta;\delta)\cap\Omega$ such that $\gamma(0)=z_n$ and $\gamma(1)=w_n$. Now $F(\gamma(t))$ must intersect the real imaginary axis. Then

$$\begin{aligned} P &= \text{last } F(\gamma(t))\cap\{y=0\} \\ Q &= 1^{\text{st}}F(\gamma(y))\cap\{x=0\}. \end{aligned}$$

Then the curve not linear $\overrightarrow{PQ}\subset$3rd quadrant. By reflectivity $\overrightarrow{PQ}$ w.r.t axis to get a closed curve γ_1 encloses a domain $U\subset\mathbb{D}$. Note that $0\in U$. Define

$$h(w)=(G(w)-\zeta)\left(\overline{G(\overline{w})}-\bar{\zeta}\right)(G(-w)-\zeta)\left(\overline{G(-\bar{w})}-\bar{\zeta}\right),\forall w\in\mathbb{D}.$$

Note h is holomorphic on $\mathbb{D}$.

Claim: $\sup_{\bar{U}}|h(w)|\leq\delta M^3$ for all $0<\delta<\frac{d}{2}$.

Assume the claim is true. Letting $\delta\to0^+$ we get that $h(w)\equiv0$. Note $|h(0)|\leq\delta M^3\to0$ which impliez $h(0)=0$. But $h(0)=|\zeta_0-\zeta|^4=d^4>0$. A contradiction.

Proof of CLaim: To see the claim $\sup_{\partial U} |h(w)| \leq \delta M^3$. On $\overrightarrow{PQ}$, We have

$$\begin{aligned} |h(w)| &\leq |G(w) - \zeta| \, M^3 \\ &= |\gamma(\theta) - \zeta| \, M^3 \\ &< \delta M^3. \end{aligned}$$

□

- Done with prelim material
- Do prelim problems from now on.
- Last homework some of Damin's favorite problems (special attention to the ones you haven't seen before, hint hint?) and old prelim problems.

4/9/2015

- Last time:

Theorem. *Let Ω be a bounded simply-connected domain in $\mathbb{D}$. If $\partial\Omega$ is a single/simple? Jordan curve, then any biholomorphic $f:\Omega\to\mathbb{D}$ can extend to a homeomorphism $\overline{\Omega}$ onto $\overline{\mathbb{D}}$.*

Lemma. *1. For all $\zeta\in\partial\Omega$, we have $\lim_{z\to\zeta,z\in\Omega}f(z)$ exists and finite.*

Lemma. *2. For any $\zeta\in\partial\Omega$, define $f(\zeta)=\lim_{z\to\zeta,z\in\Omega}f(z)$. Then $f\in C^0\left(\overline{\Omega}\right)$.*

Proof. For all $\epsilon>0$, there exists $\delta>0$ such that $|f(z)-f(\zeta)|<\epsilon$, for all $z\in B(z,\delta)\cap\Omega$. For all $\zeta_1\in B(\zeta;\frac{\delta}{2})$ pick $z_n\to\zeta_1$ with $z_n\in\Omega$. Then

$$\begin{aligned}|f(\zeta_1)-f(\zeta)| &\leq |f(\zeta_1)-f(z_n)|+|f(z_n)-f(\zeta)|\\ &< \epsilon+\epsilon=2\epsilon,\end{aligned}$$

for large n, with $z_n\in B\left(\zeta_1;\frac{\delta}{4}\right)\subset B(\zeta;\delta)$. This shows that

$$|f(w)-f(\zeta)|<2\epsilon,\ \ \forall w\in B(\zeta;\delta)\cap\overline{\Omega}.$$

□

Lemma. *3. Let $f\in C^0\left(\overline{\Omega}\right)$ be given in Lemma. Then f is one-to-one.*

Proof. Incorrect Proof:
For all $\zeta_1\neq\zeta_2\in\partial\Omega$, note that

$$f\left(B(\zeta_1;\delta)\cap\Omega\right)\cap f\left(B\left(\zeta_2;\delta\right)\cap\Omega\right)=\emptyset$$

with $0<\delta\ll$ DOES NOT IMPLY that $f\left(\zeta_1\right)\neq f\left(\zeta_2\right)$. □

Theorem. *2 (Lindelof) Let $\Gamma(t)$ be a curve in $\overline{\mathbb{D}}$ such that $|\Gamma(t)|<1$, if $0\leq t<1$ and $\Gamma(1)=1$. Let g be a bounded holomorphic function on $\mathbb{D}$ such that*

$$\lim_{t\to1}g\left(\Gamma(t)\right)=\beta\in\mathbb{C}.$$

Then $\lim_{t\to1^-}g(t)=\beta$. ($\lim_{r\to1^-}g(r)=\beta$ with $r\in\mathbb{D}\cap\{Imz=0\}$).

Proof of Lemma 3.

Proof. Suppose $\zeta_1\neq\zeta_2\in\partial\Omega$ such that $f\left(\zeta_1\right)=f\left(\zeta_2\right)=e^{i\alpha}$. Replace f by $e^{-i\alpha}f$, if necessary, we can assume $f\left(\zeta_1\right)=f\left(\zeta_2\right)=1$. Can pcik a curve γ_1,γ_2 such that $\gamma_j\left([0,1)\right)\subset\Omega,\gamma_j(1)=\zeta_j$ with $j=1,2$. Let $\Gamma_j(t)=f\left(\gamma_j(t)\right)$ is curve in $\overline{\mathbb{D}}$ satisfying Theorem 2. Let $g=f^{-1}$ on $\mathbb{D}$, g is holo, bounded and

$$\lim_{t\to1}g\left(\Gamma_1\left(t\right)\right)=\lim_{t\to1^-}\gamma_1(t)=\zeta_1$$

and by Theorem 2 it implies

$$\lim_{r\to1^-}g(r)=\zeta_1.$$

But similarly

$$\lim_{r\to1^-}g(r)=\zeta_2.$$

But $\zeta_1\neq\zeta_2$ which gives us a contradiction. □

Proof of Theorem2:

Proof. We need to show for all $\epsilon > 0$, there exists $\delta > 0$, such that

$$|g(r) - \beta| < \epsilon, \quad \forall r, \delta < r < 1.$$

Can assume $\beta = 0$ (replace g by $g - \beta$) and $|g(z)| < 1$ on $\mathbb{D}$. (g by $\frac{g(z)}{\sup_{\mathbb{D}}|g|}$). Since $\lim_{t\to 1^-} g\left(\Gamma(t)\right) = 0$. There exists $\delta > 0$ with $|g\left(\Gamma(t)\right)| < \epsilon$, for all $1 - 2\delta < t < 1$. Fix $r \in (1 - \delta, 1)$. Let

$$t_1 = \max\left\{t \mid \mathrm{Re}\Gamma(t) = r\right\}$$

with $\Gamma\left([t, 1]\right) \subset \overline{\mathbb{D}} \cap \{x \geq r\}$. Reflect $E_1 = \Gamma\left([t_1, 1]\right)$ with respect to real axis to get E_2. Reflect $E_1 \cup E_2$ with respect to $\{x = r\}$ to get E_3. Then $E = E_1 \cup E_2 \cup E_3$. Let U be the bounded open sets enclosed by E. Define

$$\begin{aligned} h(z) &= [g(z)]\left[\overline{g\left(\bar{z}\right)}\right]\left[g\left(2r - z\right)\overline{g\left(2r - \bar{z}\right)}\right] \\ &= [E_1]\,[E_2]\,[E_3]\,. \end{aligned}$$

Note that h is holomorphic on U, with $|h| < 1$. (Note $\frac{\partial}{\partial \bar{z}}\overline{g(\bar{z})} = \overline{\frac{\partial}{\partial z} g\left(\bar{z}\right)} = \frac{\partial}{\partial \bar{z}} g = 0$).

Claim: $\sup_U |h(z)| < \epsilon$. If so, $r \in U$ and $h(r) = |g(r)|^4$ which implies $|g(r)| < \epsilon^{1/4}$ then for all $1 - \delta < r < 1$. Remains to show the claim. Note

$$\sup_{z\in E_1\setminus\{1\}} |g(z)| < \epsilon \implies \sup_{E_1\setminus\{1\}} |h(z)| < \epsilon.$$

Then $E_1 = \Gamma\left([t_1, 1]\right)$. Similarly, $\sup_{E\setminus\{1, 2r-1\}} |h(z)| < \epsilon$. Let $h_C(z) = h(z)\left(1 - z\right)^c\left(2r - 1 - z\right)^c$ where $c > 0$ is a constant TBA, so h_C is holo on U, and $h_C \in C^0\left(\overline{U}\right)$. Now

$$\sup_{z\in\overline{U}} |h_C| < \epsilon M^c.$$

Letting $c \to 0^+$, we have that $\sup_{z\in\overline{U}} |h(z)| < \epsilon$. This proves the claim. □

Schwarz Reflection Principle

Lemma. *(Symmetry Principle)*

Let Ω_1 and Ω_2 be two domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\partial\Omega_1 \cap \partial\Omega_2 = \gamma$ be a rectfiable curve. Let f_j be holo on Ω_j, and $f_j \in C^0\left(\Omega_j \cup \gamma\right)$, $j = 1, 2$. If $f_1 = f_2$ on γ, then

$$f(z) = \begin{cases} f_1(z) & z \in \Omega_1 \\ f_1(z) = f_2(z) & z \in \gamma \quad \textit{is} \\ f_2(z) & z \in \Omega_2 \end{cases}$$

holo on $\Omega \equiv \Omega_1 \cup \gamma \cup \Omega_2$.

Proof. Remains to show f is holo on γ. Let T be a triangle domain, $\overline{T} \subset \Omega$. Need to show that

$$\int_{\partial T} f dz = 0$$

then we can apply Morera's theorem. For $\overline{T} \subset \Omega_1 \cup \gamma$ by Cauchy-Gorsart we have that

$$\int_{\partial T} f dz = 0.$$

If $T \cap \Omega_1 \neq \emptyset$ and $T \cap \Omega_2 \neq 0$. Let $\Gamma_0 = T \cap \gamma$ and $\Gamma_j = \overline{T} \cap \Omega_j$. Then

$$\int_{\partial T} f = \int_{\Gamma_1 + \Gamma_0} f + \int_{\Gamma_2 - \Gamma_0} f = 0.$$

□

Theorem. *3. Let Ω be a domain subset of $\mathbb{H}$. Let $I = \partial\Omega \cap \{Imz = 0\}$. Let f be holo on Ω, and $f \in C^0(\Omega \cup I)$. If $f(I) \subset \mathbb{R}$, then there exists F on $\Omega_+ \cup I \cup \Omega_-$, holo, such that $F = f$ on Ω_+.*

Proof. Let

$$F(z) = \begin{cases} f(z) & z \in \Omega_+ \cup \gamma \\ \overline{f(\bar{z})} & z \in \Omega_-. \end{cases}$$

Then F is holo on Ω_-($\frac{\partial}{\partial \bar{z}}\overline{f(\bar{z})} = \overline{\frac{\partial}{\partial z} f(\bar{z})} = 0$ then or because $\frac{\partial}{\partial z} f(\bar{z}) = 0$) On I, for all $x \in I \subset \mathbb{R}$, with $f(x) = \overline{f(\bar{x})}$. □

Question 3: Let $f \in C^0(\overline{\mathbb{D}})$ f is holo on $\mathbb{D}$. If there exists an open arc $\gamma \subset \partial\mathbb{D}$ such that $f|_{\gamma} \equiv c \in \mathbb{C}$, then $f \equiv c$ on $\overline{\mathbb{D}}$.

Proof1:

Proof. Can assume $c = 0$ ($f \mapsto f - c$) Let $L_1 : \mathbb{H} \to \mathbb{D}$ wtih $\Gamma \mapsto \gamma$ and

$$z \mapsto \frac{z - i}{z + i}.$$

So $f_1 = f \circ L_1 : \mathbb{H} \to \mathbb{C}$ is holo.

Pick an x_0 and pick a small ball such that the boundary still contains Γ. Now look f_1 on that semicircle curve ($\mathbb{H} \cup B(x_0, \delta)$) on $\mathbb{H}$, and applying Scwartzreflection preinciple, we can reflect, Call it F_1. Now $F_1|_{\Gamma} \equiv 0$ so that $F_1 \equiv 0$ Thus this implies $f \equiv 0$. □

- Q1 is his favorite problem,

4/14/2015

- #2) from the homework
  - Method1: Just show that ∞ is NOT an essential singularity. Suppose ∞ is an essential singularity. Then we know that $f(z)$ can be written as
$$f(z) = \sum_{m=0}^{\infty} c_m z^m$$
with infinitenly many nonzero terms. Use an inequality to get a contradiction.
  - For all $k \in \mathbb{N}$ $\frac{f(z)}{z^k}$ has essential singularity at ∞. By Casaroti-Weirstraa theorem: there exists $\{z_l\}_{l=1}^{\infty}$ s.t.
$$\frac{f(z_l)}{z_l^k} \to \infty \text{ as } l \to \infty.$$
  - Write
$$P_n(z) + \frac{P_{n-1}(z)}{f(z)} + \cdots + \frac{P_1(z)}{f(z)^{n-1}} = 0$$
  - Write
$$P_n(z) = a_{n,N} z^N + a_{n,N-1} z^{N-1} + \cdots + a_{n,0}.$$
  - Then
$$1 + \frac{a_{n,N-1}}{a} + \cdots + \frac{a_{n,0}}{z^N} + \sum_{j=1}^{n-1} \frac{P_j(z)}{a_{n,N} z^N f(z)} = 0. \qquad (2)$$
  - Pick $k = \max\{\deg P_j(z); 1 \le j \le n-1\} - N$. Let $k = \max\{d, 0\}$. Apply (2) to z_l. Letting $l \to +\infty$ we get that
$$1 + 0 + \sum \frac{z_l^k}{f(z_l)} = 0$$
which implies $1 = 0$, a contradiction.
  - Remark: Avoid $\infty \pm \infty$, use inequality.
  - Method2: (Tim Smits, etc)
    * Assune, without loss of generality, that there exists $R_0 > 1$ such that
$$\max_{|z| \le R} |f(z)| > 1.$$
Choose a sufficiently large $R > R_0$ such that
$$\begin{aligned} 1 &\le \delta_j |z|^{d_j} \\ &\le |P_j(z)| \\ &\le C_j |z|^{d_j}, \text{ for all } |z| = R \end{aligned}$$
Indeed
$$\begin{aligned} |P_j(z)| &= \left| a_{j,m} z^m + a_{j,m-1} z^{m-1} + \cdots + a_{j,0} \right| \\ &= |a_{j,m} z^m| \left| 1 + \frac{a_{j,m-1}}{z} + \cdots + \frac{a_{j,0}}{z^m} \right| \\ &\ge |a_{j,m}| R^m (1 - \epsilon). \end{aligned}$$
Here C_j, δ_j depdends only onthe leading terms of P_j , $d_j = \deg P_j$.

∗ Let $|z_0| \le R$ such that

$$|f(z_0)| = \max_{|z|\le R} |f(z)| \quad > 1$$

By max principle, $|z_0| = R$. So

$$\delta_j R^{d_j} \le |P_j(z_0)| \le c_j R^{d_j}, \quad 1 \le j \le n-1.$$

Now (1) $\implies$

$$\begin{aligned} |P_n(z_0) f(z_0)| &= \frac{|P_{n-1}(z_0)|}{|f(z_0)|} + \frac{|P_{n-2}(z_0)|}{|f(z_0)|^2} + \cdots + \frac{|P_1(z)|}{|f(z_0)|^{n-1}} \\ &\le |P_{n-1}(z_0)| + \cdots + |P_1(z_0)|. \end{aligned}$$

so

$$\delta_n R^{d_n} |f(z_0)| \le \sum_{j-1}^{n-1} c_j R^{d_j}$$

which implies

$$|f(z_0)| \le \sum_{j=1}^{n-1} \frac{c_j}{\delta_n} R^{d_j - d_n}$$

which implies

$$\max_{|z|\le R} |f(z)| \le CR^d$$

where $d = \max\{d \mid 1 \le j \le n\} - d_n \ge 1$.

∗ By Cauchy Integral formula

$$\begin{aligned} \left|f^{(k)}(0)\right| &= \left|\frac{k!}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta\right| \\ &\le C\frac{\max_{|\zeta|=R R^{k+1}} |f(\zeta)|}{} \\ &\le CR^{d-k-1}, \end{aligned}$$

for $k \ge d$ letting $R \to \infty$ we get that $f^{(k)}(0) = 0$. Hence

$$f(z) = \sum_{k=0}^{d} c_k z^k.$$

is a polynomial.

- #3 from the homeowor: Given

$$|f'(z)| \left(1 - |z|^2\right) + |f(0)| \le 1. \qquad (3)$$

Show that $\mathcal{F}$ is normal.

Proof. From (3) we have that

$$|f'(z)| \le \frac{1 - |f(0)|}{1 - |z|^2} \le \frac{1}{1 - |z|^2}.$$

For all compact set $K \subset \mathbb{D}$,

$$\sup_K |f(z)| \le ?$$

there exist $r \in (0,1)$ such that $K \subset B_r$. On B_r, we have that

$$|f'(z)| \leq \frac{1}{1-r^2}, \text{ for all } |z| \leq r.$$

Then

$$\begin{aligned} |f(z)| &= \left| \int_{[0,z]} f'(z) dz \right| \\ &\leq \frac{r}{1-r^2} \leq \frac{1}{1-r^2}, \end{aligned}$$

Hence

$$\sup_K |f(z)| \leq \sup_{B_r} |f(z)| \leq \frac{1}{1-r^2}.$$

□

Rest of the Course:

- 1. Picard's Little Theorem
- 2. Picard's Big Theorem
- $3^\star$ Runge's approximation Theorem

Theorem. *(PLT) If f is an entire function which omits two values, then*

$$f \equiv \textit{constant}.$$

- Remark $f(z) = e^z$ only omits 0, and its not a constant.
- Our proof uses differetial geometry and maximum principle. Due to Alphors (1938).
- Let us first consider the Scwarz lemma.
- The new idea for proving Schawrz lemma. The key observation is $\mathbb{D}$ admits a nice metric

$$\frac{idz \wedge d\bar{z}}{\left(1-|z|^2\right)^2} \equiv \omega_P, \quad \text{Poincare}$$

- Notation: $hdz \otimes d\bar{z} \sim \frac{hidz \wedge d\bar{z}}{z} \sim |h|\,|dz|^2$.
- The term

$$\frac{1}{\left(1-|z|^2\right)^2} = h$$

  satisfies the proerty

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log h = 2h \quad (\star)$$

  or using Laplace operator

$$\Delta \log h = 8h.$$

- Proof of $(\star)$ from $\log h = -2\log\left(1-|z|^2\right)$ get

$$\frac{\partial}{\partial \bar{z}} \log h = \frac{-z}{1-|z|^2}.$$

Note $|z| = z\bar{z}$ then

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \log h = \frac{-1}{1-|z|^2} + \frac{-z\bar{z}}{\left(1-|z|^2\right)^2} = \frac{-1}{\left(1-|z|^2\right)^2}.$$

- **<u>Exercise:</u>** Compute $\frac{\partial^2}{\partial z \partial \bar{z}} \log \frac{1}{1+|z|^2}$.
- Can't do the rest. Refer to Bobby's notes.

4/16/2015

Last time:

- Schwarz Lemma: If $f:\mathbb{D}\to\mathbb{D}$ holo, then
$$\frac{|f'(z)|^2}{\left(1-|f(z)|^2\right)^2}\le\frac{1}{\left(1-|z|^2\right)^2}$$
i.e. $f^\star\omega_P\le\omega_P$.

Proof. (Ahlfors. 1938) Let $F=\frac{f^\star\omega_P}{\omega_P}=\frac{\left(1-|z|^2\right)^2|f'(z)|^2}{\left(1-|f(z)|^2\right)^2}$. Then $F\ge 0$, $F\in C^\infty(\mathbb{D})$. For all $z\in\mathbb{D}$ s.t. $F(z)>0$ ($\iff$ $|f'(z)|>0$).

Then
$$\begin{aligned}\partial_z\partial_{\bar z}\log F &= \partial_z\partial_{\bar z}\log\left(1-|z|^2\right)^2+\partial_z\partial_{\bar z}\log|f'(z)|^2\\ &\quad -\partial_z\partial_{\bar z}\log\left(1-|f(z)|^2\right)^2\\ &= \partial_z\partial_{\bar z}\log\left(1-|z|^2\right)^2+0-\partial_z\partial_{\bar z}\log\left(1-|f(z)|^2\right)^2\\ &= \frac{2}{\left(1-|z|^2\right)^2}-2\partial_z\left[\frac{-\overline{f'(z)}f(z)}{1-|f(z)|^2}\right]\\ &= \frac{-2}{\left(1-|z|^2\right)^2}+\frac{2|f'(z)|^2}{\left(1-|f(z)|^2\right)^2}\\ &= \frac{2}{\left(1-|z|^2\right)^2}\left[F(z)-1\right].\end{aligned}$$

If F attains $\sup_{\mathbb{D}}F$ at $z_0\in\mathbb{D}$ then so is $\log F$. Also $\Delta\log F\le 0$. $(\Delta=4\partial_z\partial_{\bar z})$ which implies $F(z_0)\le 1$.

That is, $\sup_{\mathbb{D}}F\le 1$.

The subtle point is $\sup_{\mathbb{D}}F$ need NOT be attained in $\mathbb{D}$. Two ways to overcome this.

1 (Ahlfors): Let $\tilde F=\frac{\left(r^2-|z|^2\right)|f'(z)|^2}{r^2\left(1-|f(z)|^2\right)^2}$. That is, use metric
$$\frac{4r^2}{\left(r^2-|z|^2\right)^2}\frac{idz\wedge d\bar z}{2}\text{ on }\{|z|<r\}.$$

To show
$$\frac{|f'(z)|^2}{\left(1-|f(z)|^2\right)^2}\le\frac{r^2}{\left(r^2-|z|^2\right)^2}.$$

Letting $r\to 1^-$, yields the proof. The advatange of using $\tilde F$ is, there exists $z_2\in B(0;r)$ such that $\tilde F(z_2)=\sup_{|z|<r}\tilde F$. Then $\frac14\Delta\log\tilde F=\frac{r^2}{\left(r^2-|z|^2\right)^2}\left[\tilde F-1\right]$. At z_2, we have that
$$\Delta\log\tilde F\le 0\implies\tilde F(z)\le F(z_2)\le 1.$$

2. Yau (1978)

If (where before $\Delta \log F \geq \left(\frac{1}{1-|z|^2}(F-1)\right)$) $\Delta_P \log F \geq F-1$ where Δ_P is the Laplacian of a compelte metric space then $\sup F < 1$ then

$$\Delta = \partial^2_{xx} + \partial^2_{yy} = \partial_z \partial_{\bar{z}}.$$

□

- Schaws Lemma: If $f : B(0;R) \to U$ be holo function, U domain with a metric ω satisfying $K(\omega) \leq -\kappa < 0$ where $\kappa \equiv$ constant > 0, then

  $$f^\star \omega \quad \leq \tfrac{1}{\kappa} \quad \omega_P \text{ on } B(0;R)$$

  where $\omega_P = \frac{4R^2}{\left(R^2-|z|^2\right)^2} \frac{-dz\wedge d\bar{z}}{2}$.
  - Remark: If $\omega = g\frac{idz\wedge d\bar{z}}{2}$, define $K(\omega) = \frac{2\partial_z\partial_{\bar{z}}(-\log g)}{g}$. (This is the Gauss curvature). Recall for $\mathbb{D}$, $\omega_P = \frac{4}{\left(1-|z|^2\right)^2}\frac{idz\wedge d\bar{z}}{2}$, where $K(\omega_P) \equiv -1$.

Proof. Let $F = \frac{f^\star \omega}{\omega_P} = \frac{\left(R^2-|z|^2\right)^2 g(f(z))\left|f'(z)\right|^2}{R^2}$. Then $\Delta \log F = \frac{R^2}{\left(R^2-|z|^2\right)^2}[\kappa F - 1]$ which implies (by Ahlfors/You) that $F \leq \frac{1}{\kappa}$. □

Corollary. *2. If $f : \mathbb{C} \to U$ is holo, and U admits a metric ω with $K(\omega) \leq -\kappa < 0$, then $f \equiv const.$*

Proof. For any $R > 0$,

$$g(f(z))\,|f'(z)| \leq \frac{1}{\kappa}\frac{R^2}{\left(R^2-|z|^2\right)^2}.$$

Let $R \to +\infty$ which goes to zero, which implies $|f'(z)| \equiv 0$. □

- **Remark1:** Coro2 implies Liuville's Theorem. If $f : \mathbb{C} \to B(0;L)$ with $\frac{L^2}{\left(L^2-|z|^2\right)^2}\frac{idz\wedge d\bar{z}}{2}$, which means it has negative curvature.
  - (2) $\mathbb{C}$ does not have metric with negative curvature. $K(\omega)$
  - But $\frac{4dz\wedge d\bar{z}}{\left(1+|z|^2\right)^2}$ is a metric on $\mathbb{C}$ with $K > 0$.
    - ∗ Negative curvature is a space that looks the gabriels horn.Triangle (angles)add up to less than π. Positive curvature is a space that looks like a sphere, triangle angles add up to more than π.
      - · Negagtive curvature looks like its bloated in, while posive curvature is like bloated out.
      - · zero curvature is a flat space.
  - (3) $\mathbb{C}^\star = \mathbb{C}\backslash\{0\}$ No metric with negative curvature $\mathbb{C} \to \mathbb{C}^\star$ with e^z.

Theorem. *(Picard's Little Theorem) Any holo $f : \mathbb{C} \to \mathbb{C}\backslash\{a,b\}$ is a constant.*

Proof. We need to construct a metric ω on $\mathbb{C}\backslash\{a,b\}$ s.t. $K(\omega) \leq -\kappa < 0$.

Idea: Find a metric ω on $\mathbb{D}^\star = \mathbb{D}\backslash\{0\}$ with negative curvature, that is $K(\omega) \leq -\kappa < 0$. Get a covering map e^{iz}.

With $\omega_{P^\star} = \frac{4dz\wedge d\bar{z}}{|z|^2\log^2|z|^2}$ on $0 < |z| < 1$.

We claim $K(\omega_{P^\star}) \equiv -1$.

Proof:

$$\begin{aligned}
\partial_z\partial_{\bar z}\log\frac{1}{|z|^2\left(\log|z|^2\right)^2} &= -\partial_z\partial_{\bar z}\log|z|^2\,(\text{goes to }0) - 2\partial_z\partial_{\bar z}\log\left(\log|z|^2\right)\\
&= -2\partial_z\left[\frac{1}{\log|z|^2}\frac{z}{|z|^2}\right]\\
&= -2\partial_z\left[\frac{1}{\bar z\log|z|^2}\right]\\
&= \frac{2}{|z|^2\log^2|z|^2}.
\end{aligned}$$

∎

Claim:

4/21/2015

- 3 questions from last two problem set. One or two from his favorite questions.
- Question2: Very easy question.
  - Hint: We know $\sqrt{z}$ is not a nice function. But $\sqrt{z^2}$ always has an analytic branch on ANY domain. The reason, because $\sqrt{z^2}$ goes back to the absliute value of the number.
  - Indeed $\sqrt{z^2} = e^{\frac{1}{2}\log z^2}$ if $z = re^{i\theta}$ then $z^2 = r^2 e^{2i\theta}$ then $\log z^2 = \log r^2 + 2i\theta$ so that $\sqrt{z^2} = e^{\log r + i\theta} = re^{i\theta}$.
  - If you take $\theta \mapsto \theta + 2k\pi$ does no change since $e^{i\theta} = e^{i(\theta+2k\pi)}$ so $\sqrt{z^2} = re^{i\theta}$then we can take .
    - the baosutely value thing, take a branch cut.
    - If equal to zero then define it to be zero. $\sqrt{z^2} = 0$ if $|z| = 0$. You can also use limit to show holo at 0, or use info that we arelady know since its holo around it.
  - Also $\sqrt{h(z)}$ always has a branch in $B(0;\delta)$. If $h(0) \neq 0$, Indeed, choose $\delta > 0$ s.t. $h\,|_{B(0;\delta)} \neq 0$. A smaller disk $B(0;\delta)$ simply connected, then by Riemman mapping theorem then $\sqrt{h}$ has a branch. Check your notes.
  - In general, $h(0) = 0$ use Taylor expansion.
- **Picard's Little Theorem:** Any entire function $f : \mathbb{C} \to \mathbb{C} \setminus \{a, b\}$ is constant.

Proof. Reduces to construct a metric ω_1 on $\mathbb{C}\setminus\{0,1\}$ (Replace f by $\frac{f(z)-a}{b-a}$). A metric such that $K(\omega_1) < -\delta < 0$ with $(\delta > 0)$, so negative curvature.

Let the metric be in the following form:

$$\begin{aligned}
\omega_1 &= \left\{ \left(1+|z|^2\right)^2 |z|^2 \left[\log \frac{|z|^2}{1+|z|^2}\right]^2 \right. \\
&\quad \left. \cdot \left(1+|z-1|^2\right)^2 |z-1|^2 \left[\log \frac{|z-1|^2}{1+|z-1|^2}\right]^2 \right\}^{-1} \\
&\quad \cdot \frac{idz \wedge d\bar{z}}{2} \\
&\equiv g_1 \cdot g_2 \cdot \frac{idz \wedge d\bar{z}}{2}.
\end{aligned}$$

where

$$\begin{aligned}
g_1^{-1} &= \left(1+|z|^2\right)^2 |z|^2 \left[\log \frac{|z|^2}{1+|z|^2}\right]^2, \\
g_2^{-1} &= \left(1+|z-1|^2\right)^2 |z-1|^2 \left[\log \frac{|z-1|^2}{1+|z-1|^2}\right]^2.
\end{aligned}$$

So

$$\begin{aligned}
K(\omega_1) &\equiv -\frac{2}{g_1 g_2} \partial_z \partial_{\bar{z}} \log(g_1 g_2) \\
&= -\frac{2}{g_1 g_2} \left(\partial_z \partial_{\bar{z}} \log(g_1) + \partial_z \partial_{\bar{z}} \log(g_2)\right) \\
&= \frac{2}{g_2} \left(-\frac{1}{g_1} \partial_z \partial_{\bar{z}} \log(g_1)\right) - \frac{2}{g_1} \left(-\frac{1}{g_2} \partial_z \partial_{\bar{z}} \log(g_2)\right).
\end{aligned}$$

Only need to compute the Gauss curvature of $-\frac{2}{g_1}\partial_z\partial_{\bar z}\log(g_1)$ by letting $z\mapsto z-1$. Note that $\partial_z\partial_{\bar z}\log|z|^2=0$ for all $z\neq 0$. And so

$$\partial_z\partial_{\bar z}\log\left(1+|z|^2\right)^2=\frac{2}{\left(1+|z|^2\right)^2}.$$

Then

$$\partial_z\partial_{\bar z}\log\left(\log\frac{|z|^2}{1+|z|^2}\right)^2=\frac{1}{\left(1+|z|^2\right)^2|z|^2\left[\log\frac{|z|^2}{1+|z|^2}\right]^2}$$

by noting that $\frac{|z|^2}{1+|z|^2}=1-\frac{1}{1+|z|^2}$. Then

(1) $K(\omega_1)<0$ on $\mathbb{C}\setminus\{0,1\}$

(2) Also $\lim_{z\to 0}K(\omega_1)=-8[\log 2]^2=\lim_{z-1}K(w_2)$

(3) $\lim_{z\to\infty}K(\omega_1)=-4$. Which implies that $\infty<-\delta_2\le K(\omega_1)\le-\delta_1$.

This finishes the proof. □

- Homework Problem3: At most one connected component. Use Picard's Little Theorem
  - Because it is vey precise. The entire function can only omit possibly one point.

Theorem. *2 (Picard's Big Theorem) Let f be holo on $B^\star(a;\delta)$. If f has an essential singularity at a, then $\forall 0<\delta_1<\delta$ we have that*

$$f\left(B^\star(a;\delta_1)\right)=\mathbb{C}\ \textit{or}\ \mathbb{C}\setminus\{\textit{one point}\}.$$

- This is an extensino of the Casorati-Weisrtrass Theorem.
- **Remark**: PBT $\Longrightarrow$ CW and PBT $\Longrightarrow$ PLT.
- **Fact 1:** Let f be a meromorphic function on $\Omega\subset\mathbb{C}$. Then f cna be viewed as a holomorphic function with values in $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, the Riemann sphere. Indeed, let $z=a$ be a pole of f. Then we note that locally f can be written as the following form

$$f(z)=\frac{h(z)}{(z-a)^m}\text{ in }B^\star(a;\delta),$$

  where h is holo on $B(a;\delta)$. Then $f:B(a;\delta)\to\hat{\mathbb{C}}$. Let $I(z)=\frac{1}{z}$ for all $z\neq 0$. Then $(I\circ f)$ is holo on $B(a;\delta)$.
- **Fact 2:** There is a metric ω_s on $\hat{\mathbb{C}}$ such that $\omega_s\mid_{\mathbb{C}}=\frac{idz\wedge d\bar z}{2\left(1+|z|^2\right)^2}$ and $K(\omega s)\equiv+1$ on $\hat{\mathbb{C}}$.
  - With $\mathbb{C}=\hat{\mathbb{C}}\setminus\{\infty\}$ Near ∞ let $w=\frac{1}{z}$ then $[w=0\leftrightsquigarrow z=\infty]$ and $z=\frac{1}{w}$ and $\bar z=\frac{1}{\bar w}$ then lets compute the metric: We have $dz=-\frac{1}{w^2}dw$ and $d\bar z=-\frac{1}{\bar w^2}d\bar w$ and $|z|^2=\frac{1}{|w|^2}$ Then

$$\frac{idz\wedge d\bar z}{2\left(1+|z|^2\right)^2}=\frac{idw\wedge d\bar w}{2\left(1+|w|^2\right)^2}.$$

    This is invariant. That means this is always of the form

$$\omega_s\sim\frac{idz\wedge d\bar z}{2\left(1+|z|^2\right)^2}\nearrow\frac{2dz\otimes d\bar z}{\left(1+|z|^2\right)^2}$$

    for every $z\in\hat{\mathbb{C}}$. Then take the square root of this guy:

$$d_S(0,z)=\int_0^{|z|}\frac{dr}{1+r^2}=2\tan^{-1}|z|\le\pi.$$

Definition. Let $\mathcal{F}$ be a family of meromorphic functions on a domain $\Omega \subset \mathbb{C}$. We say $\mathcal{F}$ is normal, if for all $\{f_n\} \subset \mathcal{F}$, has convergent subsequence $\{f_{n_k}\}$ with respect to d_S such that

$$d_S\left(d_{n_k}, g\right) \to 0 \text{ as } k \to +\infty$$

(or replace with convergence uniformly on every compact subset) where g is a meromorphic function on Ω.

Theorem. *(Marty's criterion)*

Let $\mathcal{F}$ be a family of meromorphic functions on Ω. Then $\mathcal{F}$ is normal $\iff$ for every compact $K \subset \Omega$, we have that

$$f^{\star}\omega_S$$

is uniformly bounded for every $f \in \mathcal{F}$.

This is equivalent to: For every compact $K \subset \Omega$,

$$\exists M_K > 0, \quad \frac{|f'(z)|}{1 + |f(z)|^2} \leq M_K, \ \ \forall f \in \mathcal{F}.$$

Proof. Lets first prove ($\Longleftarrow$): By $A - A$ lemma, for all $\{f_n\} \subset \mathcal{F}$, we have that $\{f_n\}$ is equibounded, and

$$d_S\left(f_n, 0\right) \leq \pi, \quad \forall n.$$

To check $\{f_n\}$ is equi-continuous on K. Then for all $z_1 \in K$, pick $B(z_1;\delta) \subset \Omega$, for all $z_2 \in B(z_1,\delta)$ we have that

$$\begin{aligned} d_S\left(f_n\left(z_1\right), f_n\left(z_2\right)\right) &\leq \int_{f(\gamma)} \omega_S = \int_{\gamma} f^{\star}\omega_S \\ &\leq M_K\left|z_1 - z_2\right|, \end{aligned}$$

which shows this family is equicontinuous. Then apply Arzella-Ascoli's Theorem. □

4/23/2015

- $\hat{\mathbb{C}}$ - The Riemann sphere.

Proposition. *$Aut\left(\hat{\mathbb{C}}\right) = SL\left(2,\mathbb{C}\right) \setminus \{\pm I_2\}$. Here*

$$SL\left(2,\mathbb{C}\right) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C}, ad-bc=1 \right\}.$$

Proof. Let $\varphi \in \text{Aut}\left(\hat{\mathbb{C}}\right)$. If $\varphi\left(\infty\right) = \infty$, then $\varphi \in \text{Aut}\left(\mathbb{C}\right)$ which implies $\varphi(z) = az + bz$ for $a, b \in \mathbb{C}$.

Now if $\varphi\left(\infty\right) \neq \infty$, let $\psi(z) = \frac{1}{\varphi(z)-\varphi(\infty)}$ then $\psi\left(\infty\right) = \infty$ and $\psi \in \text{Aut}\left(\mathbb{C}\right)$. Which implies that $\psi(x) = cz + d$. But then

$$\begin{aligned} \varphi(z) &= \frac{1}{cz+d} + \varphi(\infty) \\ &= \frac{az+b}{cz+d}, \end{aligned}$$

by letting $a = c\varphi(\infty)$ and $b = d\varphi(\infty) + 1$. Let $\mathcal{F} : \text{SL}\left(2;\mathbb{C}\right) \to \text{Aut}\left(\hat{\mathbb{C}}\right)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d},$$

and notice that $\ker \mathcal{F} = \{\pm I_2\}$. Since its surjective and a homomorphism then First isomorphicm theroem says that $\text{Aut}\left(\hat{\mathbb{C}}\right) \cong \text{SL}(2;\mathbb{C}) / \left\{\pm I_2\right\}$. ☐

- Remark: We have that $\text{Aut}\left(\hat{\mathbb{C}}\right) \cong GL\left(2;\mathbb{C}\right) / \left\{\lambda I_2; \lambda \in \mathbb{C} \setminus \{0\}\right\}$.

Proposition. *2. We have d_S on $\hat{\mathbb{C}}$ with*

$$\begin{aligned} \omega_S &= \frac{4idz \wedge d\bar{z}/2}{\left(1+|z|^2\right)^2} \\ &= \frac{4dz \otimes d\bar{z}}{\left(1+|z|^2\right)^2}, \end{aligned}$$

For every $z, w \in \hat{\mathbb{C}}$ we have that

$$d_S\left(z,w\right) = \inf_{\gamma} \int_{\gamma} \frac{2\left|dz\right|}{\sqrt{1+|z|^2}},$$

and

$$\begin{aligned} d_S\left(z,w\right) &= \int_{\gamma} \frac{2\left|dz\right|}{\sqrt{1+|z|^2}}, \gamma\text{=great circle passing thoughz}, w \\ &= 2\tan^{-1}\left|\frac{z-w}{1+\bar{w}z}\right| \qquad (1). \end{aligned}$$

- One way to see (1). Assume $w=0$ and get that

$$d_S(z,0)=\int_0^{|z|}\frac{2dr}{1+r^2}=2\tan^{-1}|z|.$$

- If $w\neq 0$ let $\varphi(z)=\frac{z-w}{1+\bar{w}z}\in \operatorname{Aut}\left(\hat{\mathbb{C}}\right)$. Note that

$$\begin{aligned}\varphi^{\star}\omega_S &= \frac{4|\varphi'(z)|^2\,dz\otimes d\bar{z}}{\left(1+|\varphi(z)|^2\right)^2}\\ &= \frac{4dz\otimes dz}{\left(1+|z|^2\right)^2}.\end{aligned}$$

- Then

$$\begin{aligned}d_S(z,w) &= d_S(\varphi(z),\varphi(w))\\ &= d_S(\varphi(z),0)\\ &= 2\tan^{-1}|\varphi(z)|\\ &= 2\tan^{-1}\left|\frac{z-w}{1+\bar{w}z}\right|.\end{aligned}$$

Theorem. *(Marty's criterion) Let $\mathcal{F}$ be a family of meromorphic functions on $\Omega\subset\mathbb{C}$. Then $\mathcal{F}$ is normal $\iff$ for every compact $K\subset\Omega$, there exists $M_K>0$ such that*

$$\sup_{f\in\mathcal{F}}\sup_{z\in K}\frac{|f'(z)|}{1+|f(z)|^2}\leq M_K.$$

Proof. ($\Longleftarrow$) Done last class.

($\Longrightarrow$) Suppose NOT. Then there exists a K such that and $\{f_k\}\subset\mathcal{F}$ such that $\frac{|f_k'(z)|}{1+|f_k(z)|^2}$ is unbounded on K. Since $\mathcal{F}$ is normal. Then $\{f_k\}\overset{d_S}{\rightrightarrows} g$ on K. If $\sup_K|g|<\infty$, by Weirstrass, g is holo on K, and $f_k\rightrightarrows g$. Now

$$\frac{|f_k'(z)|}{1+|f_k(z)|^2}\to\frac{|g'(z)|}{1+|g(z)|^2}$$

which the thing on the right is bounded on K. If $g(z_0)=\infty$, for some $z_0\in K$. Pick $\overline{B(z_0;\delta)}\subset\Omega$, such that $\frac{1}{g}$ is bounded on $\overline{B(z_0;\delta)}$, so is $\frac{1}{f_{n_k}}$ for large k. By Weiretrass, we have that $\frac{1}{g}$ is holomorphic and $\frac{1}{f_{n_k}}\rightrightarrows\frac{1}{g}$. But

$$\frac{\left|\left(\frac{1}{g}\right)'(z)\right|}{1+\left|\frac{1}{g}\right|^2}\leftarrow\frac{\left|\left(\frac{1}{f_{n_k}}\right)'(z)\right|}{\left|1+\left(\frac{1}{f_{n_k}}\right)^2(z)\right|}=\frac{\left|f_{n_k}'(z)\right|}{1+|f_{n_k}(z)|^2}\text{ bounded on.}$$

∎

Lemma. *2. Let $\mathcal{F}$be a family of meromorphic functions on Ω. If for every $f\in\mathcal{F}$ then*

$$f(\Omega)\subset\hat{\mathbb{C}}\backslash\{P,Q,R\}$$

where P,Q,R are 3 distinct points, then $\mathcal{F}$ is normal.

Proof. Can assume $\{P,Q,R\} = \{0,1,\infty\}$ otherwise, $\varphi(z) = (z,P,Q,R) = \frac{z-P}{z-R}\frac{Q-R}{Q-P}$. Now $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, replace f by $\varphi \circ f$. so $f(\Omega) \subset \mathbb{C}\setminus\{0,1\}$. Note it suffices to show $\sup_{f\in\mathcal{F}} f^{\star}\omega_S$ is bounded on each $B(z_0;R)$ and $\overline{B(z_0,R)} \subset \Omega$. Recall in the proof of PLT, we construct a metric ω on $\mathbb{C}\setminus\{0,1\}$. Now

$$\begin{aligned}\omega &= \left\{\left(1+|z|^2\right)^2 |z|^2 \left[\log\frac{|z|^2}{1+|z|^2}\right]^2\right.\\ &\quad \left.\cdot\left(1+|z-1|^2\right)^2 |z-1|^2\left[\log\frac{|z-1|^2}{1+|z-1|^2}\right]^2\right\}^{-1}\\ &\quad \cdot\frac{idz\wedge d\bar{z}}{2}\end{aligned}$$

with $K(\omega) \leq -\delta < 0$. Note $\omega_S = \frac{4idz\wedge d\bar{z}/2}{\left(1+|z|^2\right)^2}$. We have

$$\begin{aligned}\frac{\omega_S}{\omega} &= \left\{\left(1+|z|^2\right)^2 |z|^2 \left[\log\frac{|z|^2}{1+|z|^2}\right]^2\right.\\ &\quad \left.\cdot\left(1+|z-1|^2\right)^2 |z-1|^2\left[\log\frac{|z-1|^2}{1+|z-1|^2}\right]^2\right\}\\ &\leq C \text{ on } \mathbb{C}\setminus\{0,1\}.\end{aligned}$$

where

$$\log\frac{|z|^2}{1+|z|^2} = \log\left[1-\frac{1}{1+|z|^2}\right] \sim -\frac{1}{1+|z|^2} \text{ as } |z|\to\infty$$

using $\log(1-x) \leq x$. And similarly

$$\log\frac{|z-1|^2}{1+|z-1|^2} \sim \frac{-1}{1+|z-1|^2},$$

as $z\to\infty$. This implies that $\omega_S \leq C\omega$, which implies that

$$\begin{aligned}f^{\star}\omega_S &\leq Cf^{\star}\omega\\ &\leq \frac{C}{\delta}\frac{4R^2}{\left(R^2-|z|^2\right)^2}\frac{2dz\wedge d\bar{z}}{2}.\end{aligned}$$

for every $B(0;r) \subset B(0;R)$. So

$$\sup_{f\in\mathcal{F}} f^{\star}\omega_S \leq C\frac{R^2}{\left(R^2-r^2\right)^2} \text{ on } B(0;R),$$

which finishes the proof. □

Theorem. *(Picard's big theorem) If f has an essential singularity at $z=a$ and f is holomorphic on $B^{\star}(a;\delta)$ then for every δ_1 satisfying $0<\delta_1<\delta$, we have*

$$f\left(B^{\star}(a;\delta_1)\right) = \mathbb{C} \textit{ or } \mathbb{C}\setminus\{\textit{one point}\}.$$

Proof. Suppose NOT. Then there exists a $\delta_1 < \delta$ such that
$$f\left(B^{\star}\left(a;\delta_1\right)\right) \subset \mathbb{C}\backslash\left\{b,c\right\}.$$
Can assume $\{b,c\} = \{0,1\}$ by considering the map $f \mapsto \frac{f(z)-b}{c-b}$. Can ssume $a = 0$ by considerig the map $z =\mapsto z - a$. So we have
$$f\left(B^{\star}\left(a;\delta_1\right)\right) \subset \mathbb{C}\backslash\left\{0,1\right\} = \hat{\mathbb{C}}\backslash\left\{0,1,\infty\right\}.$$
Let for every $n \in \mathbb{N}$ be $f_n(z) = f\left(\frac{z}{n}\right)$ for $0 < |z| < \delta_2$. The $\mathcal{F} \equiv \{f_n\}$ omits $\{0, 1, \infty\}$. By Lemma 2, $\mathcal{F}$ is normal. This means that there exists subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that
$$f_{n_k} \overset{d_S}{\rightrightarrows} g$$
uniformly on every compact $K \subset B^{\star}(0;\delta_1)$. Let $K = \left\{|z| = \frac{\delta_1}{2}\right\}$. If $\sup_{z\in K}|g(z)| < \infty$, then g is holo on K. We have $f_{n_k} \rightrightarrows g$ on K. Now $f_{n_k}(z) = f\left(\frac{z}{n_k}\right)$ is bounded for large k, which implies
$$|f(z)| \leq M, \forall\, |z| = \frac{\delta_1}{2n_k}.$$
By maximum principle,
$$|f(z)| \leq M, \quad \forall 0 < |z| < \delta_1/2$$
which implies $z = 0$ is a removable singularity, which is a contradiction. If $g\left(z_0\right) = \infty$ for some $z_0 \in K$, pick $\overline{B\left(z_0;\delta_0\right)} \subset \Omega$ such that $\frac{1}{g}$ is holo. We also have $\frac{1}{f_{n_k}} \to \frac{1}{g}$ on $\overline{B\left(z_0,\delta\right)}$. A contradiction. $\square$

4/28/2015

- Homeowor Problems Set 6
- #1, If $g:\Omega\to\mathbb{D}$ and $\Omega\subset\mathbb{D}$ then take g^{-1}.
- #2: Consider $\sqrt{f}$ has a branch on $B(0,\delta)$if $f(0)\neq 0$. If $f(0)=0$ then $f(z)=z^m h(z)$ on $B(0;\delta)$.
  - Now $f(z^2)=z^{2m}h(z^2)$ because $\sqrt{f(z^2)}=\sqrt{z^{2m}h}$.
  - Now z^m is a branch of $\sqrt{z^{2m}}$. Take $z^m h$.
- #3, Should be $\mathbb{C}\backslash\{f(z)\mid f(z)<M\}$ Problem 7, 2010 Aug. and this is $\{|w|\geq M\}\setminus\{\text{one point}\}$.
  - The statement is Not true! $\mathbb{C}\backslash\{z\mid |f(z)|\leq M\}$ is NOT true.
  - Example: $f(z)=\sin z$ and $|f|^2=\sinh^2 y+\sin^2 x$ and then $\{|f(z)|\leq 2\}$ contains a strip which exlcudes the real axis. So $\mathbb{C}\backslash\{z\mid |f(z)|<M\}$ has two components.
  - Remark1: The statemeent is true if $\mathbb{C}$ is replaced by $\hat{\mathbb{C}}$. The point: $\hat{\mathbb{C}}\backslash\{z\mid |f(z)|<M\}$ each conpoene tof is unbounded.
  - Remark2: The original statement is true if f is a polynomial.
    - More "generally", we say a function f is proper if f^{-1} (compact set in $\mathbb{C}$) is compact in $\mathbb{C}$.
    - Prop: If f is proper, then $\mathbb{C}\backslash\{z\mid |f(z)|<M\}$ has at most one component.
  - Remark3: An entire function is proper if and only if f is a polynomial.
    - (Weisrtrass) $\iff$ For every unbounded $\{z_n\}$, $\{f(z_n)\}$ is unbounded.
    - Idea: If $\mathbb{C}\backslash\{z\mid |f(z)|<M\}$ has two components $E_1\bigsqcup E_2$. Then one of them say E_1, has to be bounded. Then apply max princinple to E_1 and get a contradiction.
- #4, Apply the downward box to e^{-z^2} and get $e^{-\frac{a^2}{4b^2}}\sqrt{\pi}$.
  - Original integral $\int_{-\infty}^{\infty}e^{iax-bx^2}dx$.
- #5, Use

$$F(z)=\begin{cases}f(z) & z\in\mathbb{D}\\ \frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}} & z\notin\mathbb{D}.\end{cases}$$

On $\partial\mathbb{D}$ we have $|f(z)|=1$ and using $\frac{1}{\bar{z}}=z$ get $f(z)=\frac{1}{\overline{f(z)}}$ and so its well defined on $\partial\mathbb{D}$.

  - Now must verify this is holomorphic:

$$\begin{aligned}\frac{\partial}{\partial\bar{z}}\left(\frac{1}{\overline{f\left(\frac{1}{\bar{z}}\right)}}\right) &= \frac{1}{\left(\bar{f}\right)^2}\left(-\frac{\partial}{\partial\bar{z}}\bar{f}\right)\\ &= \frac{1}{\left(\bar{f}\right)^2}\left(-\overline{\frac{\partial}{\partial z}f\left(\frac{1}{\bar{z}}\right)}\right)\\ &= \frac{1}{\left(\bar{f}\right)^2}(-0)=0,\end{aligned}$$

    if $f\left(\frac{1}{\bar{z}}\right)\neq 0$. N ow this happens if and only if $f(z)\neq 0$ in $z\in\mathbb{D}$. and $\frac{1}{\bar{z}}$ and z are symmetric with respect to $\partial\mathbb{D}$.
  - Now for all$z\in\partial\mathbb{D}|f(z)|=1$ implies f has finitely many zeros unless $f\equiv e^{ia}$.
  - Then appply symmetry princinples on a smaller circle inside $\mathbb{D}$ that contains all the zeros and apply it there.
  - Now $f(a_1(=0$ implies $F\left(\frac{1}{\bar{a_1}}\right)=\frac{1}{\overline{f(a_1)}}=\infty$ which implies F has poles at $a_1,\ldots,a_N$.
- #5b) With $F(z)=e^{i\alpha}\prod_{j=1}^{N}\frac{z-a_j}{1-\bar{a_j}z}$. Let $h(z)=\prod_{j=1}^{N}\frac{F(z)(1-\bar{a_j}z)}{z-a_j}$. Then $h(z)$ has removable singularity at $a_1,\ldots,a_m$. Now consider $\frac{1}{\bar{a_1}},\ldots,\frac{1}{\bar{a_m}}$.

  - Then h is holo on $\mathbb{C}$. Since $|h|_{\partial\mathbb{D}} \equiv 1$ and h is nowhere zero in $\mathbb{D}$. which implies $h \equiv c = e^{ia}$ in $\mathbb{D}$ on $\mathbb{C}$.
- #6, By Montel's Theorem, the sequence $\{f_k\}$ we have that $\{f_{k_l}\} \rightrightarrows_{cpt} g$ on Ω. Now $E = \{z \mid \lim f_k = f\} \subset \{z \mid g = f\}$ has a limit point. Thus $f_{k_l} \rightrightarrows_{cpt} f$. We need to show the whole sequence converges uniformly to f, not just the subsequence.
  - Method1(Prove by contradiction): We have that for every compact $K \subset \Omega$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$\sup_{z\in K} |f_k(z) - f(z)| < \epsilon, \quad \forall k \geq N.$$
  - There exists a compact $K \subset \Omega$ compact, $\exists \epsilon, \forall N \in \mathbb{N}, \exists k \geq N$ such that
$$\sup_{z\in K} |f_k(z) - f(z)| \geq \epsilon$$
which implies there exists $\{z_l\}_{l=1}^{\infty}$ such that $|f_{k_l}(z_l) - f(z_l)| \geq \epsilon$. Since K is compact $\{z_{l_m}\} \to z_\star \in K$ by Bolzona Weistrass, Thus there exists
$$\{f_{k_{l_m}}\} \rightrightarrows f$$
by Montel's again, but this is a contradiction of the epsilon.
  - Method2: It sufccies to show $E = \Omega$. For every $K \subset \Omega$ compact let $K = \bigcup_{j=1}^{N} B(a_j; \delta)$ for finitely many. For $1 \leq j \leq N$ we get
$$\sup_{1\leq j\leq N} |f_k(a_j) - f_l(a_j)| < \epsilon \quad k, l \geq L$$
for every $z \in K$, we have that $z \in B(a_j; \delta)$ and choosing δ small enough we get
$$\begin{aligned} |f_k(z) - f_l(z)| &\leq |f_k(z) - f_k(a_j)| + |f_k(a_j) - f_l(a_j)| + |f_l(a_j) - f_l(z)| \\ &< 3\epsilon. \end{aligned}$$
Now $\{f_k\}$ Cauchy sequence on K and $f_k \rightrightarrows f$. To show $E = \Omega$, suppose $z_0 \in \Omega \backslash E$. Montel's, there exists $f_{n_k}(z_0) \to A \neq f(z_0)$. Now $\left\{f_{n_{k_l}}\right\} \rightrightarrows_K g$ and $g = f$ on Ω. Then
$$A = \lim f_{n_{k_l}} = g(z_0) = f(z_0)$$
which is a contradiction.
- #7: If $\int_{\gamma_1} f = \int_{\gamma_2} f$ If yes, then there exists $F = \int_{z_0}^{z} f$ such that $F' = f$.
  - Now
$$\int_{\gamma_1 - \gamma_2} f = \oint_{\Gamma} f = \sum_{i=1}^{p} \text{Res}(f, a_i) = 0$$
by the second problme is not equal to zero.
- Final Exam:
  - 3 problems, haven't yet made it
  - 1. problem from the last two homework sets. HW5/6
  - 1. problem from his favorite questions.
    * Q1 punturee disk
    * Q2 first problem in HW6
    * Q3 If I give you a holomorhic funciton on the disk, if there exist an arc on the circle γ (arc only on a part of the circle) with $f\,|_\gamma \equiv c$ then shjow that $f \equiv c$ on $\overline{\mathbb{D}}$.
      · Use Swarzat reflection princinple. Consider $\mathbb{H}$ extend, thus f has to identically zero. simialr to problem, its like problem 5, just use symmetry princinple.

      * Q+ Residue(no an original favorite problem)
   - 1 from a previous prelim, one haven't done yet on HW, like one where you can use Swarz lemma.
- 5121, Cpx2, 5411 PDE
- 5030 differential geometry.