On quantitative absolute continuity of harmonic measure and big piece approximation by chord-arc domains

Steve Hofmann (joint work with J. M. Martell)

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F. and M. Riesz (1916): Ω ⊂ ℂ, simply connected. Then ∂Ω rectifiable implies ω ≪ σ.

C.E. due to C. Bishop and P. Jones (1990): conclusion need not hold w/o some connectivity.

Notation: ω = harmonic measure (at generic point in Ω), σ = ℋ^{1}|_{∂Ω} (or σ = ℋ^{d−1}|_{∂Ω} in ℝ^d).

Recall: ∂Ω rectifiable = covered by a countable union of Lipschitz graphs, up to a set of ℋ^1 (or ℋ^{d−1}) measure 0.
What about higher dimensions? (note: \( d = n + 1 \) from now on)

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Remark: it follows that Dirichlet problem solvable with \( L^p \) data, some \( p < \infty \) (in fact, in Lip domain can take \( p = 2 \) or even \( 2 - \varepsilon \)).
$A_\infty$ more precisely:

- $\omega \in A_\infty(\sigma)$ means that $\forall B$ centered on $\partial \Omega$ with $r_B < \text{diam}(\partial \Omega)$, and $\forall$ Borel $E \subset \Delta := B \cap \partial \Omega$, $X \in \Omega \setminus 4B$

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\omega^X(E) \lesssim \left( \frac{\sigma(E)}{\sigma(\Delta)} \right)^\theta \omega^X(\Delta).
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- weak-$A_\infty$ is the same but with $\omega^X(2\Delta)$ on RHS.

I.e., weak-$A_\infty$ is $A_\infty$ but w/o doubling.

- Note that $A_\infty$ and weak-$A_\infty$ are each quantitative, scale invariant versions of absolute continuity.
David-Jerison (1990), and independently Semmes: $\Omega$
“chord-arc” domain (aka CAD) in $\mathbb{R}^{n+1}$, then $\omega \in A_\infty(\sigma)$.

Definition: CAD = NTA + ADR boundary

ADR: $\sigma(\Delta(x, r)) \approx r^n$

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CS: $\exists B' \subset B \cap \Omega$, with $r_{B'} \approx r_B$; denote by $X_B =$ center of $B'$; this is a “CS point relative to $B$”.

HC: quantitative scale invariant path connectedness.
Method of proof of [DJ]: ADR + 2-sided CS implies “Interior Big Pieces of Lipschitz Sub-Domains” (IBPLSD); i.e., for every $B$ centered on $\partial \Omega$, with $r_B < \text{diam}(\partial \Omega)$, $\exists$ subdomain $\Omega_B \subset \Omega \cap B$ s.t.

- $\Omega_B$ is a Lipschitz domain, with constants uniform in $B$. 

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Q: why does this give $A_\infty$?

- IBPLSD implies: by Dahlberg (applied in $\Omega_B$), plus maximum principle, obtain $\exists \eta \in (0, 1)$ s.t. for Borel $E \subset \Delta$,

\[
\sigma(E) \geq (1 - \eta)\sigma(\Delta) \implies \omega^{X_B}(E) \gtrsim 1.
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- Then use pole change formula for harmonic measure (uses HC), to change scales, i.e., to improve to $\omega \in A_\infty(\sigma)$. 
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w/o HC, pole change formula unavailable; [BL] argument “changes pole w/o pole change formula”, this (necessarily) introduces errors which result in non-doubling; weak-$A_\infty$ is best possible conclusion.
Some Converse results:

- Lewis - Vogel (2007): $\partial \Omega$ ADR, $\omega \approx \sigma$; i.e., $k := \frac{d\omega}{d\sigma} \approx 1$ (after normalizing). Then $\partial \Omega$ is Uniformly Rectifiable (UR) (quantitative scale invariant version of rectifiability - David-Semmes).

Proof idea (both papers), based on Alt-Caffarelli technique: small oscillation of $\nabla G$ plus non-degeneracy of $\nabla G$ implies flatness.
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Recent Results (posted late 2017- early 2018)

- J. Azzam: \( \partial \Omega \) ADR, then

\[ \omega \in A_{\infty}(\sigma) \iff \partial \Omega \text{ UR and } \Omega \text{ “semi-uniform” (S-U)}. \]

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Proof ingredients:

- $\omega$ doubling $\iff \Omega$ is S-U (improved Aikawa result).
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- $\omega \in A_\infty \implies \partial \Omega$ UR by S.H. - Martell.
- UR + S-U implies IBPCAD; so, get (*) by M.P. + [DJ], improve to weak-$A_\infty$ by [BL], then S-U gives doubling, hence $A_\infty$. 
Remark: note that connectivity in Azzam’s result (S-U condition) is about doubling, not about absolute continuity.

OTOH, in light of Bishop-Jones example, the question remains: what is minimal connectivity assumption, which, in conjunction with UR, yields quantitative absolute continuity of harmonic measure?
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- Combining work of two different groups of authors, we can now answer this.
Let \( \Omega \subset \mathbb{R}^{n+1} \) be an open set with interior CS, and ADR boundary. Then TFAE:

1. \( \partial \Omega \) is UR, and \( \Omega \) satisfies “Weak Local John” (WLJ) condition.
2. \( \Omega \) satisfies Interior Big Pieces of Chord-Arc Domains (IBPCAD).
3. \( \omega \in \text{weak-}A_\infty(\sigma) \).

WLJ entails connected non-tangential path from CS point \( X_B \) to a “big piece” portion of \( \Delta = B \cap \partial \Omega \); (could also be thought of as “Weak Local S-U”).
Recent Results (continued)

Evolution of this result:

1. \( (1) \implies (2) \) new result of S.H. - Martell

Remark: direct proof \( (1) \implies (3) \) is slightly earlier result (a few months ago) of S.H. - Martell.

Remark: background hypotheses (upper and lower ADR, interior CS are in nature of best possible - \( \exists \) C.E. in absence of any one of them.)
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- $(1) \implies (2)$ new result of S.H. - Martell
- $(2) \implies (3)$ immediate from M.P. plus [DJ] plus [BL] as described above.

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- (3) $\implies$ (1) has two parts: weak-$A_\infty$ $\implies$ UR is S.H. - Martell result mentioned earlier; weak-$A_\infty$ $\implies$ WLJ is new result of Azzam-Mourgoglou-Tolsa.

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Proof ingredients:

- (1) $\Rightarrow$ (2): Corona approximation of UR set by CAD’s (S.H. - Martell - Mayboroda 2016) plus 2-parameter bootstrapping scheme based on “extrapolation of Carleson measures” (J. Lewis).

- (3) $\Rightarrow$ (1): (new part of [AMT]) use of Alt-Caffarelli-Friedman monotonicity formula to establish connectivity.
Thank you!