Dimension of a certain measure associated to p-Laplace type operator

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Outline

Part I - Dimension of a certain measure in the plane

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Fix p, $1 < p < \infty$ and suppose that u is p-harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, \mathrm{d}x = 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega \cap N).$$

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Assume that $u > 0$ in $\Omega \cap N$ and $u = 0$ on $\partial\Omega$ in the Sobolev sense.

Set $u \equiv 0$ in $N \setminus \Omega$. Then $u \in W^{1,p}(N)$.

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Set $u \equiv 0$ in $N \setminus \Omega$. Then $u \in W^{1,p}(N)$.

It is well know from [HKM, Chapter 21] that there is a finite, positive, Borel measure μ_p associated with u satisfying

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \psi \rangle \, \mathrm{d}x = -\int \psi \, \mathrm{d}\mu_p \quad \text{for all nonnegative } \psi \in C_0^\infty(N).$$

μ_p has support on $\partial\Omega$ and is called p-harmonic measure.

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Let $\mathcal{H}^\lambda(E)$ denote the Hausdorff measure of $E \subset \mathbb{R}^2$ relative to λ defined in the following way;

for fixed $0 < \delta < r_0$ let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \;\; i = 1, 2, ...$

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Set $\phi_\delta^\lambda(E) := \inf_{L(\delta)} \sum \lambda(r_i)$. Then $\mathcal{H}^\lambda(E) := \lim_{\delta\to 0} \phi_\delta^\lambda(E)$.

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$$\mathcal{H} - \dim \nu := \inf\{\alpha \,|\, \exists \text{ a Borel set } E \subset \partial\Omega; \ \mathcal{H}^\alpha(E) = 0, \ \nu(\mathbb{R}^2 \setminus E) = 0\}.$$

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When everything is smooth,

$$\mathrm{d}\mu_p = |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^1|_{\partial\Omega}.$$

A measure μ is said to be absolutely continuous with respect to another measure ν if for every Borel set $E \subset \partial\Omega$ with $\nu(E) = 0$ then we have $\mu(E) = 0$. In this case we use the following notation

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A set E is said to have $\sigma-$finite ν measure if

$$E = \bigcup_{i=1}^{\infty} E_i$$

with $\nu(E_i) < \infty$ for $i = 1, \ldots, \infty$.

Results of interest in the plane for harmonic measure

When $p = 2$ then we have the usual Laplace equation. In this case, if u is the Green's function for Laplace's equation with pole at some $z_0 \in \Omega$, then the measure associated with this function u is harmonic measure, $\omega(\cdot, z_0)$.

[C]: Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113123, 1985.

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Theorem (Carleson in [C])

$\mathcal{H} - dim\, \omega = 1$ when $\partial\Omega$ is a snowflake in the plane and $\mathcal{H} - dim\, \omega \leq 1$ when Ω is the complement of a self similar Cantor set.

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Theorem (Makarov in [M])

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. Then

a) *$\omega \ll \mathcal{H}^\lambda$ where $\lambda(r) := r \exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$ if A is large.*

b) *ω is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$ measure.*

Therefore, $\mathcal{H} - \dim\,\omega = 1$ when $\Omega \subset \mathbb{R}^2$ is simply connected.

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For $1 < p \neq 2 < \infty$, we have the $p-$harmonic measure, μ_p, associated with a p-harmonic function u.

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Theorem (Bennewitz and Lewis in [BL])

If $\partial\Omega$ is a quasi circle in the plane then $\mathcal{H} - dim\, \mu_p \geq 1$ when $1 < p < 2$ while $\mathcal{H} - dim\, \mu_p \leq 1$ if $2 < p < \infty$. Moreover, strict inequality holds for $\mathcal{H} - dim\, \mu_p$ when $\partial\Omega$ is the Von Koch snowflake.

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Theorem (Lewis, Nyström, and Poggi-Corradini in [LNP])

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain and let $\hat{\lambda}(r) := r \exp\{A\sqrt{\log 1/r \log\log 1/r}\}$.

a) $\mu_p \ll \mathcal{H}^{\hat{\lambda}}$ *when* $1 < p < 2$ *for some* $A = A(p) \geq 1$.

b) μ_p *is concentrated on a set of* $\sigma-$*finite* $\mathcal{H}^{\hat{\lambda}}$ *when* $2 < p < \infty$ *for some* $A = A(p) \leq -1$.

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Theorem (Lewis in [L])

Let

$$\lambda(r) := r \exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$$

and let $\Omega \subset \mathbb{R}^2$ *is a bounded simply connected domain.*

a) *If* $1 < p < 2$, *then* $\mu_p \ll \mathcal{H}^\lambda$ *for* $A = A(p)$ *sufficiently large.*
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b) *If $2 < p < \infty$, then μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$.*

$$\mathcal{H} - \dim \mu_p \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

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Combining results of Makarov and Lewis we see

$$\mathcal{H} - \dim \mu_p \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ = 1 & \text{when } p = 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

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Is there any other measure or PDE that one can study the same problem?

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In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\text{div}\, \mathcal{A}(x, \nabla u) = 0$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

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If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

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In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

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h is called $\delta-$monotone for some $0 < \delta \leq 1$ in $\mathbb{R}^2$ if $h \in W^{1,1}(B(0,R))$ for each $R > 0$ and $\langle h(x) - h(y), x - y \rangle \geq \delta |h(x) - h(y)| \, |x - y|$ for a.e. $x, y \in \mathbb{R}^2$.

Let p be fixed and $1 < p < \infty$ and let f be a function with;

(a) $f : \mathbb{R}^2 \to (0, \infty)$ is homogeneous of degree p.

That is, $f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0$ when $\eta \in \mathbb{R}^2 \setminus \{0\}$.

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(b) f is uniformly convex in $B(0,1) \setminus B(0,1/2)$.

That is, $\exists c \geq 1$ such that for a.e. $\eta \in \mathbb{R}^2$, $\frac{1}{2} < |\eta| < 1$ and all $\xi \in \mathbb{R}^2$ we have $c^{-1}|\xi|^2 \leq \sum_{j,k=1}^{2} \frac{\partial^2 f}{\partial \eta_j \eta_k}(\eta)\xi_j\xi_k \leq c|\xi|^2$.

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Examples for such f;

- $f(\eta) = |\eta|^p$ for $1 < p < \infty$.
- $f(\eta) = |\eta|^p(1 + \epsilon\eta_1/|\eta|)$ for small $\epsilon > 0$.

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where v is in a certain subclass of the Sobolev space $W^{1,p}$. Then $u > 0$ is a weak solution to the Euler Lagrange equation in $\Omega \cap N$;

$$\triangle_f u := \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial \eta_k}(\nabla u) \right) = \sum_{j,k=1}^{2} f_{\eta_k \eta_j}(\nabla u) u_{x_j x_k} = 0.$$

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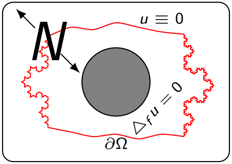
Assume that u has zero continuous boundary values on $\partial\Omega$. Extend u to all N by setting $u \equiv 0$ in $N \setminus \Omega$ in the Sobolev sense.



There is a finite, positive, Borel measure μ_f with support on $\partial\Omega$ satisfying

$$\int \langle \mathrm{D}f(\nabla u), \nabla \phi \rangle \mathrm{d}x = -\int \phi \, \mathrm{d}\mu_f \text{ whenever } \phi \in C_0^\infty(N) \text{ and } \phi \geq 0$$

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If $\partial\Omega$ and ∇u are smooth enough then

$$\begin{aligned}\int \langle \mathrm{D}f(\nabla u), \nabla\phi\rangle \mathrm{d}x &= -\int \phi \langle \mathrm{D}f(\nabla u), \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^1 \\ &= -p\int \phi \frac{f(\nabla u)}{|\nabla u|}\mathrm{d}\mathcal{H}^1.\end{aligned}$$

Therefore when $\partial\Omega$ and ∇u are smooth enough,

$$\mathrm{d}\mu_f = p\frac{f(\nabla u)}{|\nabla u|}\,\mathrm{d}\mathcal{H}^1|_{\partial\Omega}.$$

Theorem (Akman in [A14])

Let $\hat{\lambda} := r\exp\{A\sqrt{\log 1/r \log\log 1/r}\}$ *and* $\Omega \subset \mathbb{R}^2$ *be a bounded simply connected domain and* N *be a neighborhood of* $\partial\Omega$*. Let* f *be as above and let* $u > 0$ *be a weak solution to* $\triangle_f u = 0$ *in* $\Omega \cap N$ *with continuous zero boundary values on* $\partial\Omega$*. Let* μ_f *be the measure associated with* u*.*

a) *If* $1 < p \leq 2$*, there exists* $A = A(p, f) \geq 1$ *such that* $\mu_f \ll \mathcal{H}^{\hat{\lambda}}$*.*
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This result is analogue of Lewis, Nyström, and Poggi-Corradini's result under this generalized setting. It is weaker than Makarov's result when $p = 2$ and Lewis's result for other p because of $\hat{\lambda}$.

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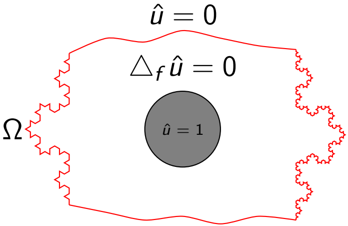
$\hat{u}$ is called capacitary function for D if $\hat{u}$ is positive weak solution to $\triangle_f \hat{u} = 0$ in D with continuous boundary values $\hat{u} = 1$ on $\partial \overline{B(z_0, d(z_0, \partial\Omega)/2)}$ and $\hat{u} = 0$ on $\partial\Omega$.



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1. $\hat{u}_z$ is a $K-$quasiregular mapping.

   i.e., $\hat{u}_z \in W^{1,2}$ locally and $|\hat{u}_{\bar{z}}| \leq k|\hat{u}_z|$ a.e. in Ω, and $k = (K-1)/(K+1)$ where

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial x_2}\right) \text{ and } \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}\right)$$

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4. $\zeta = \hat{u}$, $\zeta = \hat{u}_{x_k}$, $k = 1, 2$, are solution to $L\zeta = 0$. Moreover, $\log f(\nabla \hat{u})$ is a super solution when $1 < p < 2$, a solution when $p = 2$, and a sub solution when $2 < p < \infty$ to $L\zeta$ where

$$L\zeta := \sum_{i,j=1}^{2} (f_{\eta_i \eta_j} \zeta_{x_j})_{x_i}.$$

Define

$$w(z) = \begin{cases} \max(v(z), 0) & \text{when } 1 < p < 2 \\ \max(-v(z), 0) & \text{when } 2 < p < \infty \end{cases}$$

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Using the fundamental inequality, sub/super solution estimates, and an induction argument, we get

Lemma

Let m be a nonnegative integer. Then there exists $c_* = c_*(f, p) \geq 1$ such that for $0 < t < 1/2$,

$$\int_{\{z \in D:\ \hat{u}(z) = t\}} w^{2m} \frac{f(\nabla \hat{u})}{|\nabla \hat{u}|} \mathrm{d}\mathcal{H}^1 \leq c_*^{m+1} m! [\log \frac{1}{t}]^m.$$

Then the result follows from this Lemma and measure theoretic arguments.

Current research project

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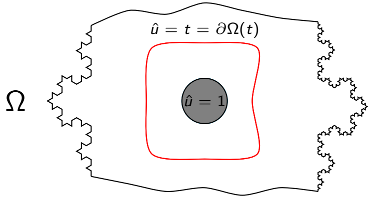

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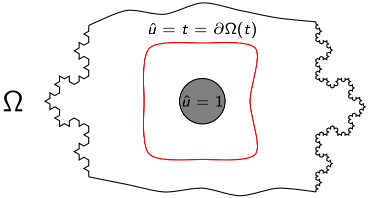

As $\partial\Omega(t)$ is a smooth curve we then have

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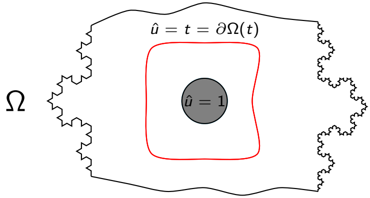

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Moreover,

$$\hat{\mu}_f^t(\partial\Omega(t)) = p\int_{\partial\Omega(t)}\frac{f(\nabla\hat{u})}{|\nabla\hat{u}|}d\mathcal{H}^1 = \xi > 0 \text{ and } \xi \text{ independent of } t \in (0,1].$$

The Law of the iterated Logarithm for certain functions

When $f(\eta) = |\eta|^2$*, i.e., under the harmonic setting, Makarov proved that if* $\phi : \mathbb{D} \to \Omega$ *is a conformal mapping then*

$$\limsup_{r\to 1} \frac{|g(r\xi)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \leq C$$

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holds almost every $\xi \in \partial\mathbb{D}$ *where* $g = \log(\phi')$.

When $f(\eta) = |\eta|^p$, *i.e., under the p-harmonic setting, Lewis proved that*

$$\limsup_{t\to 0} \frac{w(\sigma(\hat{z}, 1-t))}{\sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}} \leq c = c(p)$$

for almost every $\hat{z} \in \partial\Omega(t_0)$ *with respect to a certain measure where* $w = \max(\log|\nabla u| - c, 0)$ *and* $\sigma(\hat{z}, \cdot)$ *is trajectories orthogonal to the levels of* u *with* $\sigma(\hat{z}, 1-t) \to \partial\Omega$ *as* $t \to 0$.

As

$$\triangle_f \hat{u} = \sum_{j=1}^{2} (f_{\eta_j}(\nabla \hat{u}))_{x_j} = \sum_{j,k=1}^{2} f_{\eta_j \eta_k}(\nabla \hat{u}) \hat{u}_{x_k x_j} = 0.$$

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If we set

$$v_{x_1} = -f_{\eta_2}(\nabla \hat{u}) \quad \text{and} \quad v_{x_2} = f_{\eta_1}(\nabla \hat{u})$$

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We can also show that v is a solution to the following quasilinear elliptic equation

$$\triangle_f v = \sum_{j,k=1}^{2} f_{\eta_j \eta_k}(\nabla \hat{u}) v_{x_k x_j} = 0$$

As $f_{\eta_i \eta_j}$ are bounded and uniformly elliptic then v_z is also K-quasiregular and $v_z \neq 0$ in D and also the fundamental inequality holds for v.

1. $\hat{u} + \mathrm{i}v$ is a K'−quasiregular mapping.

[A]: Lars V. Ahlfors. Complex analysis. McGraw-Hill Book Co., New York, third edition, 1978.

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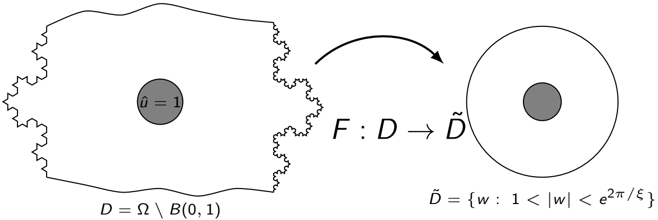
Using Stöilov factorization theorem for $\hat{u} + \mathrm{i}v$ and following [A]; F can be uniquely extended to D to get a sense preserving mapping from $D \to \tilde{D}$. Moreover, it can be shown that F is one to one and onto.



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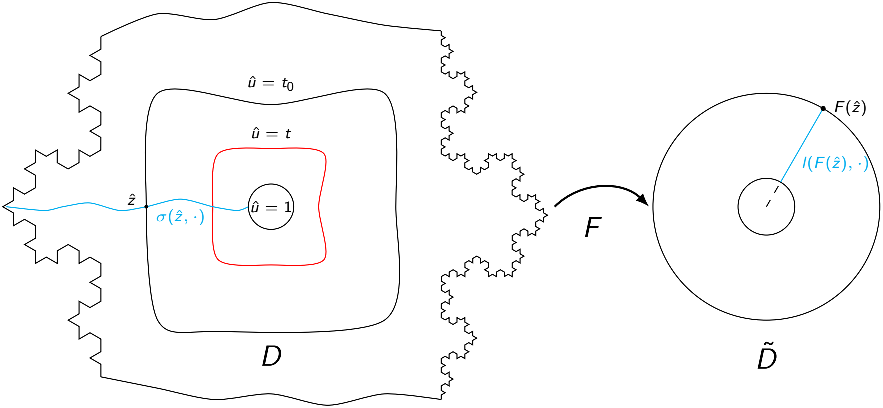

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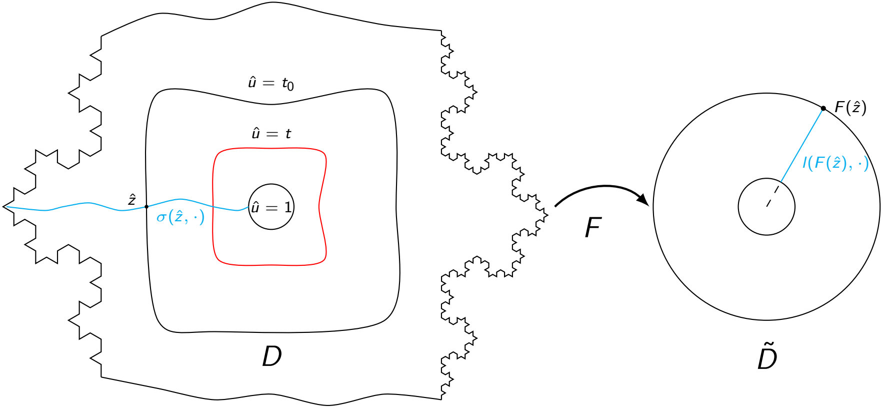

Define $\sigma(\hat{z}, \cdot) := F^{-1}(l(F(\hat{z}), \cdot))$. Then v is constant on $\sigma(\hat{z}, \cdot)$.

Existence of the curve $\sigma(\hat{z}, t)$ can also follow from the solution of ordinary differential equation;

$$\frac{\mathrm{d}\sigma(\hat{z}, t)}{\mathrm{d}t} = \frac{-\mathrm{D}f(\nabla \hat{u})(\sigma(\hat{z}, t))}{pf(\nabla \hat{u})(\sigma(\hat{z}, t))} = \left(\frac{-f_{\eta_1}(\nabla \hat{u})(\sigma(\hat{z}, t))}{pf(\nabla \hat{u})(\sigma(\hat{z}, t))}, \frac{-f_{\eta_2}(\nabla \hat{u})(\sigma(\hat{z}, t))}{pf(\nabla \hat{u})(\sigma(\hat{z}, t))} \right).$$

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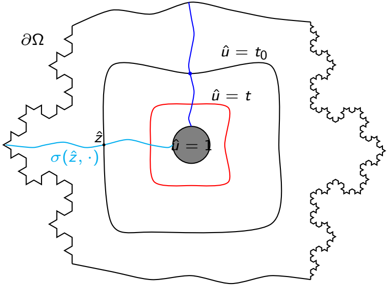
2. We also observe that v is constant along $\sigma(\hat{z}, t)$ as

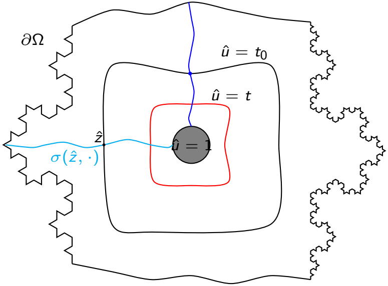
$$\frac{\mathrm{d}v(\sigma(\hat{z}, t))}{\mathrm{d}t} = \langle\nabla v(\sigma(\hat{z}, t)), \frac{\mathrm{d}\sigma(\hat{z}, t)}{\mathrm{d}t}\rangle = 0.$$



Following Lewis's work we can show that

$$\limsup_{t\to 0} \frac{w(\sigma(\hat{z}, 1-t))}{\sqrt{\log(1/t)\ \log\log\log(1/t)}} \leq \hat{c} = \hat{c}(p, f).$$

holds $\hat{\mu}_f^{t_0}$ for almost every $\hat{z}_0 \in \partial\Omega(t_0)$ where $w(z) = \max(\log f(\nabla \hat{u}) - c, 0)$ for $z \in D$ and c is chosen so that $w \equiv 0$ in $\overline{B(0,2)} \setminus B(0,1)$.



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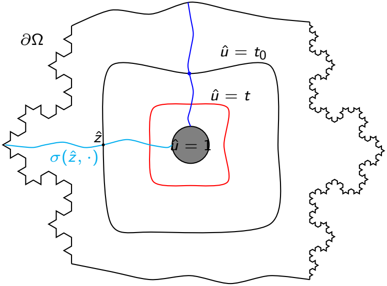

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Plausible Theorem

There exists $A = A(p, f) \geq 1$ *such that* $\hat{\mu}_f \ll \mathcal{H}^{\lambda}$ *for* $1 < p \leq 2$.

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This pde is

$$\mathcal{L}\zeta = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(b_{ij}\zeta_{x_j}) = 0$$

and $\zeta = \log|\nabla u|$ is sub/super solution in the p-harmonic setting and $\zeta = \log f(\nabla u)$ in the general setting where b_{ij} is the usual coefficients in the p-Laplace equation in the p-harmonic setting and $f_{\eta_i \eta_j}$ in the general setting.

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If one can come with such $\mathcal{L}$ for $\mathcal{A}-$harmonic PDEs then this tool can be used to study Hausdorff dimension of $\mathcal{A}-$harmonic measure in the simply connected domain in the plane.

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Let p be fixed and $1 < p < \infty$ and let f be a function with following properties;

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(b) f is uniformly convex in $B(0,1) \setminus B(0,1/2)$.

That is, Df is Lipschitz and $\exists c \geq 1$ such that for a.e. $\eta \in \mathbb{R}^n$,
$\frac{1}{2} < |\eta| < 1$ and all $\xi \in \mathbb{R}^n$ we have $c^{-1}|\xi|^2 \leq \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial \eta_j \eta_k}(\eta)\xi_j\xi_k \leq c|\xi|^2$.

Let $O \subset \mathbb{R}^n$ be an open set. Let $\hat{z} \in \partial O$ and $\rho > 0$. Let $u > 0$ be a weak solution in $O \cap B(\hat{z}, \rho)$ to

$$\triangle_f u := \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial \eta_k} (\nabla u) \right) = \sum_{j,k=1}^{n} f_{\eta_k \eta_j} (\nabla u) u_{x_j x_k} = 0.$$

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$$\int \langle \mathrm{D} f(\nabla u), \nabla \phi \rangle \mathrm{d}x = - \int \phi \, \mathrm{d}\mu_f \ \text{ whenever } \ \phi \in C_0^\infty(B(\hat{z}, \rho)).$$

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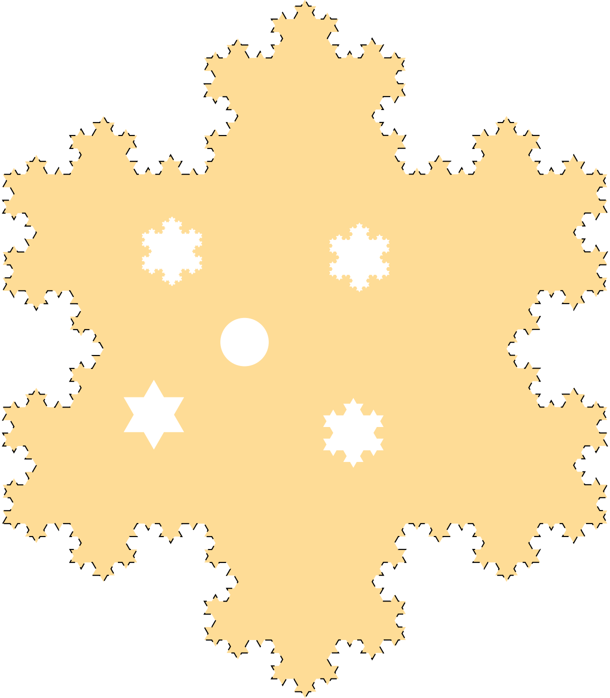
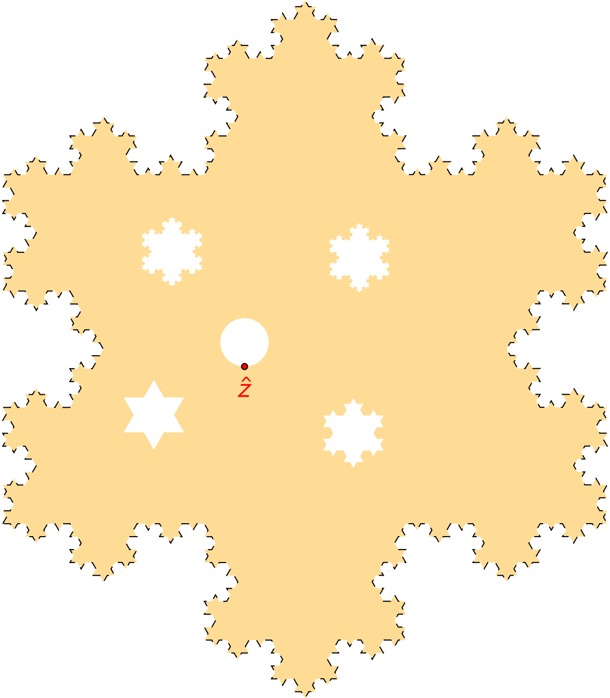
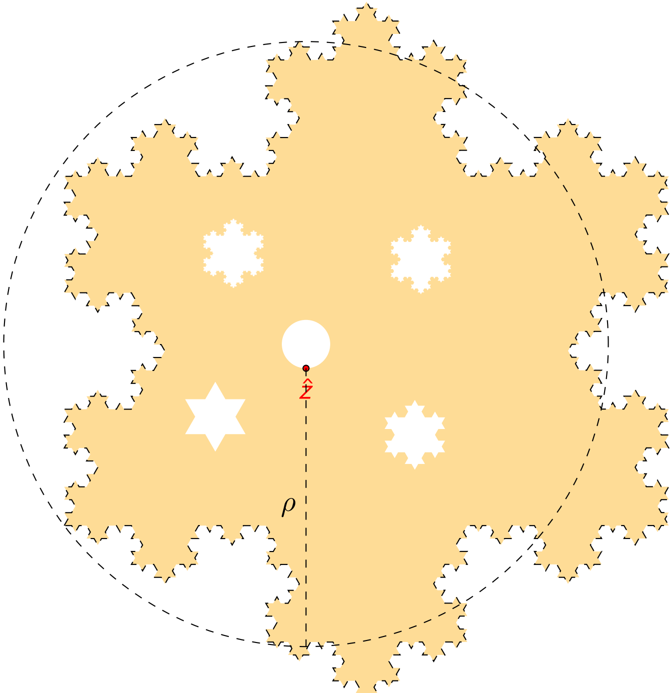
When everything is smooth enough we have

$$\mathrm{d}\mu_f = p \frac{f(\nabla u)}{|\nabla u|} \, \mathrm{d}\mathcal{H}^{n-1}|_{\partial O \cap B(\hat{z}, \rho)}.$$




$\hat{Z}$


$\hat{z}$
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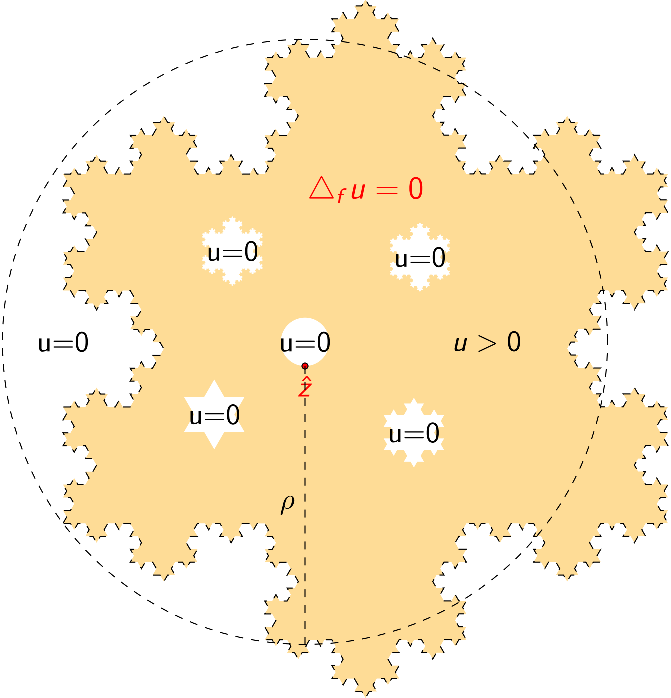

$\triangle_f u = 0$
u=0
u=0
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$u > 0$
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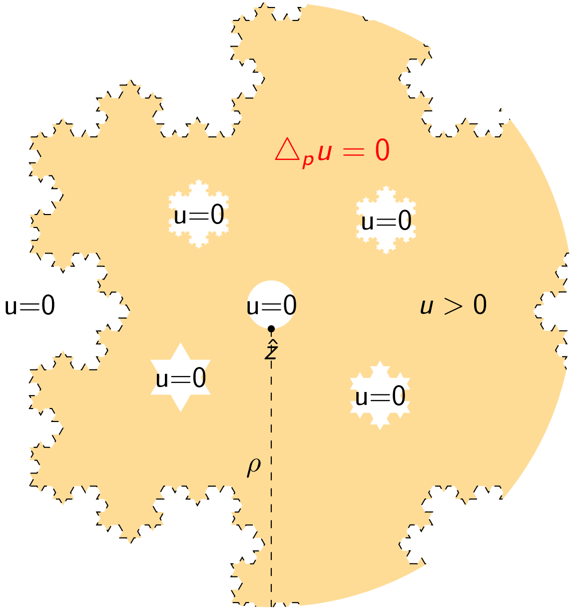

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Results of interest in space for harmonic measure, ω

When $f(\eta) = |\eta|^2$, i.e., μ_f is the usual Harmonic measure, ω, associated with u.

[JW]: Peter W. Jones and Thomas Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131144, 1988.

[W93]: Thomas Wolff. Plane harmonic measures live on sets of -finite length. *Ark. Mat.*, 31(1):137-172, 1993.

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Theorem (Bourgain in [B])

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There exists a Wolff snowflake in $\mathbb{R}^3$ for which $\mathcal{H} - dim\,\omega < 2$, and there is another one for which $\mathcal{H} - dim\,\omega > 2$.

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Theorem (Lewis, Nyström, and Vogel in[LNV])

- *μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure when $\partial\Omega$ is sufficiently "flat" and $p \geq n$.*
- *All examples produced by Wolff snowflake has $\mathcal{H} - \dim \mu_p < n - 1$ when $p \geq n$.*
- *There is a Wolff snowflake for which $\mathcal{H} - \dim \mu_p > n - 1$ when $p > 2$, near enough 2*

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We improve this result by proving

Theorem (Akman, Lewis, and Vogel in [ALV])

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$. Let $u > 0$ be p-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$, and μ_p be the p-harmonic measure associated with u.

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Therefore $\mathcal{H} - \dim \mu_p \leq n - 1$ when $p \geq n$.

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Observe that this Theorem is the natural extension of the result of Wolff to $\mathbb{R}^n$ (and Jones and Wolff's result).

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Following similar arguments from our previous result we show that

Theorem (Akman, Lewis, and Vogel in [ALV14])

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$. Let f be as above. Let $u > 0$ be a weak solution to $\triangle_f u = 0$ in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$, and μ_f be the measure associated with u.

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[ALV14]: Murat Akman, John Lewis, and Andrew Vogel, $\sigma-$finiteness of a certain measure arising from a positive weak solution to a quasilinear elliptic PDE in space. in preparation.

An example of domain in $\mathbb{R}^n$ for which $\mathcal{H}-dim\,\mu_f < n-1$

When $f(\eta) = |\eta|^2$, i.e., $\mu_f = \omega$ then there is an unpublished result;

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Let $Q_{11}, \ldots, Q_{14}$ *be the squares of the four corners of* $\mathcal{C}_0$ *of side length* a_1,

$0 < \alpha < a_1 < \beta < 1/4$, *and let* $\mathcal{C}_1 = \bigcup_{i=1}^{4} Q_{1i}$.

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Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the square of corners of each Q_{1i}, $i = 1, \ldots, 4$ of side length $a_1 a_2$, $\alpha < a_2 < \beta$. Let $\mathcal{C}_2 = \bigcup_{j=1}^{16} Q_{2j}$.

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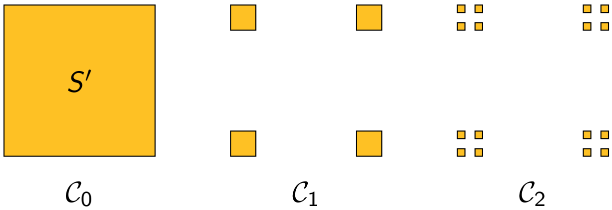
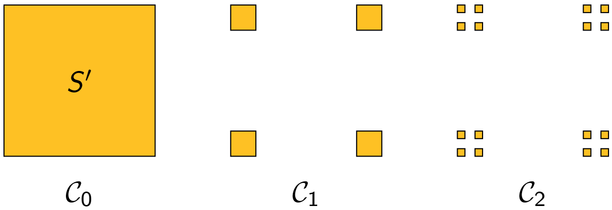
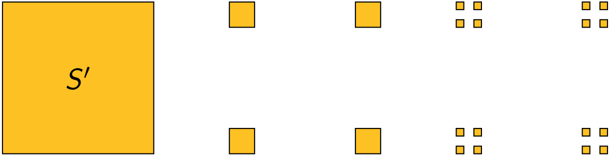

S'
$\mathcal{C}_0$
$\mathcal{C}_1$
$\mathcal{C}_2$



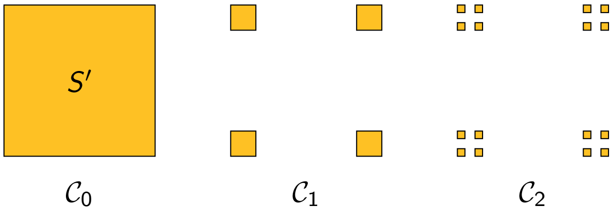
Continuing recursively, at the mth step we get 4^m squares Q_{mj}, $1 \leq j \leq 4^m$ of side length $a_1 a_2 \dots a_m$, $\alpha < a_m < \beta$ and let $\mathcal{C}_m = \bigcup_{j=1}^{4^m} Q_{mj}$.



$\mathcal{C}_0$ $\mathcal{C}_1$ $\mathcal{C}_2$

Continuing recursively, at the mth step we get 4^m squares Q_{mj}, $1 \leq j \leq 4^m$ of side length $a_1 a_2 \ldots a_m$, $\alpha < a_m < \beta$ and let $\mathcal{C}_m = \bigcup_{j=1}^{4^m} Q_{mj}$.

Then $\mathcal{C}$ is obtained as the limit in the Hausdorff metric of $\mathcal{C}_m$ as $m \to \infty$



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Then $\mathcal{C}$ is obtained as the limit in the Hausdorff metric of $\mathcal{C}_m$ as $m \to \infty$ Following Jones and Wolff arguments and using sub solution estimates we show that

Theorem (Akman, Lewis, and Vogel in [ALV14])

Let $\mathcal{S} = 2S' \subset \mathbb{R}^n$ and let u be a positive weak solution to $\triangle_f u = 0$ in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u = 1$ on $\partial\mathcal{S}$ and $u = 0$ on $\mathcal{C}$. Let μ_f be the associated measure to u.

Then $\mathcal{H} - dim\ \mu_f < n - 1$ when $p \geq n$.

THANKS!