Dimension of $p$-harmonic measure and related problems

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ICMAT

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Seminari d’edps i aplicacions

7 April 2016

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“I used to be in love with the Laplacian so worked hard to please her with beautiful theorems. However she often scorned me for the likes of Björn Dahlberg, Gene Fabes, Carlos Kenig, and Thomas Wolff. Gradually I became interested in her sister the $p$ Laplacian, $1 < p < \infty$, $p \neq 2$. I did not find her as pretty as the Laplacian and she was often difficult to handle because of her nonlinearity. However over many years I took a shine to her and eventually developed an understanding of her disposition. Today she is my girl and the Laplacian pales in comparison to her.”

— John Lewis
The size of support of $\omega$, $\mu_p$, $\mu_f$, harmonic measure, $p$-harmonic measure, Elliptic measure on rough domains in terms of the Hausdorff measure.
The size of support of \( \omega \), harmonic measure \( \mu_p \), p-harmonic measure \( \mu_f \), Elliptic measure on rough domains in terms of the Hausdorff measure.
Goals

The size of support of

\[ \begin{cases} 
\omega, \text{ harmonic measure} \\
\mu_p, \text{ p-harmonic measure} \\
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\end{cases} \]

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The size of support of \( \omega, \mu_p, \mu_f \), \( p \)-harmonic measure, Elliptic measure on rough domains in terms of the Hausdorff measure.
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The size of support of \( \omega \), harmonic measure
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\( \sim \) on rough domains in terms of the Hausdorff measure.
Outline

- Introduction

- Part I: $\sigma$—finiteness of $p$-harmonic measure in space for $p \geq n$

- Part II: Example of a domain for which $\mathcal{H} \dim \mu < n - 1$ for $p \geq n$.

- Part III: Related Work
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

Let $N$ be open neighborhood of $\partial \Omega$.

Fix $p$, $1 < p < \infty$ and suppose that $u$ is $p$-harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, dx = 0 \quad \text{for all } \phi \in W^{1,p}_0(\Omega \cap N).$$

If $u$ has continuous second partials in $\Omega \cap N$ and $\nabla u \neq 0$ then $u$ is a classical solution to the p-Laplace equation in $\Omega \cap N$:

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-4} [(p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} + |\nabla u|^2 \Delta u] = 0.$$

This is a degenerate/singular quasilinear elliptic PDE.
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• Assume $u = 0$ on $\partial \Omega$ in the Sobolev sense.

• Set $u \equiv 0$ in $N \setminus \Omega$. Then $u$ is $p$-harmonic in $N$.

By [HKM, Chapter 21] that there exists* a finite, positive, Borel measure $\mu$ associated with $u$ whose support contained in $\partial \Omega$ which satisfies

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\int \langle |\nabla u|^{p-2} \nabla u, \nabla \psi \rangle \, dx = - \int \psi \, d\mu \quad \text{for all nonnegative } \psi \in C_0^\infty(N).
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- $\mathcal{H}_\delta^\lambda(E)$ denotes the $(\lambda, \delta)$—Hausdorff content of $E$

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When $\lambda(r) = r^\alpha$ we write $\mathcal{H}^\alpha$ for $\mathcal{H}^\lambda$.

- Define the Hausdorff dimension of a Borel measure $\nu$ by

$$\mathcal{H} - \dim \nu := \inf \{ \alpha \mid \exists \text{ a Borel set } E \subset \partial \Omega; \mathcal{H}^\alpha(E) = 0, \nu(\mathbb{R}^n \setminus E) = 0 \}$$

i.e., it is the “smallest dimension” of a set with full $\nu$ measure.

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Results of interest for harmonic measure

When $p = 2$ and $u$ is the Green’s function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

- **Carleson**: $\mathcal{H} - \dim \omega = 1$ when $\partial \Omega$ is snowflake in the plane. $\mathcal{H} - \dim \omega \leq 1$ when $\partial \Omega$ is a self similar Cantor set.

- **Jones-Wolff**: $\mathcal{H} - \dim \omega \leq 1$.

- **Wolff**: $\omega$ is concentrated on a set of $\sigma$–finite $\mathcal{H}^1$ measure.

Indeed, $\omega$ lives on

$$F = \left\{ z \in \partial \Omega : \limsup_{r \to 0} \frac{\omega(B(z, r))}{r} > 0 \right\}.$$
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- \( \mathcal{H} - \dim \omega \geq n - 2 \) by an easy computation for any domain \( \Omega \subset \mathbb{R}^n \).

- **Bourgain**: \( \mathcal{H} - \dim \omega \leq n - \tau(n) \) whenever \( \Omega \subset \mathbb{R}^n \).

- **Wolff**: \( \exists \) Wolff snowflakes in \( \mathbb{R}^3 \) \( \Leftrightarrow \mathcal{H} - \dim \omega > 2 \), \( \mathcal{H} - \dim \omega < 2 \).

- **Lewis-Verchota-Vogel**: Wolff’s result holds in \( \mathbb{R}^n \); Harmonic measure on both sides of a Wolff snowflake, say \( \omega_+, \omega_- \) could have

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\max(\mathcal{H} - \dim \omega_+, \mathcal{H} - \dim \omega_-) < n - 1,
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or

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Results of interest for $p$-harmonic measure

For general $p \neq 2$;

- **Bennewitz-Lewis**: If $\partial \Omega \subset \mathbb{R}^2$ is a quasi-circle then
  \[
  \begin{align*}
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For general $p \neq 2$;

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Part I: \(\sigma\)–finiteness of p-harmonic measure in space for \(p \geq n\)

To state our recent work we need a notion of \(n\) capacity. If \(K \subset \overline{B}(x, r)\) is a compact set, define \(n\)–capacity of \(K\) as

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\text{Cap}(K, B(x, 2r)) = \inf \int_{\mathbb{R}^n} |\nabla \psi|^n dx
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where the infimum is taken over all infinitely differentiable \(\psi\) with compact support in \(B(x, 2r)\) and \(\psi \equiv 1\) on \(K\).

A compact set \(E \subset \mathbb{R}^n\) is said to be locally \((n, r_0)\) uniformly fat or locally uniformly \((n, r_0)\) thick provided there exist \(r_0\) and \(\beta > 0\) such that whenever \(x \in E\), \(0 < r \leq r_0\)

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Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

Let $u > 0$ be $p$-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$.

Extend $u$ to all $B(\hat{z}, \rho)$ by defining $u \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then $u$ is $p$-harmonic in $B(\hat{z}, \rho)$.

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Same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat.

Indeed $\mu$ lives on $\mathcal{P}$ where

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- If \( w \in \partial O \) and \( B(w, 4r) \subset B(\hat{z}, \rho) \) then there exists \( c = c(p, n) \geq 1 \) with
  \[
  \frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w, r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).
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- The left-hand side is true for any open set \( O \) and \( p \geq n \).

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The tools we have used requires to find a PDE in divergence form for which \( u, u_{x_k} \) are both solutions and \( \log |\nabla u| \) is a sub solution for \( p \geq n \) at points where \( \nabla u \neq 0 \).

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L \zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (b_{ij} \zeta_j) \quad \text{where} \quad b_{ij} = |\nabla u|^{p-4} [(p - 2) u_{x_i} u_{x_j} + \delta_{ij} |\nabla u|^2]
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Our result follows from this proposition.

**Proposition 1**

Let $\lambda$ be a non-decreasing function on $[0, 1]$ with

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\lim_{t \to 0} \frac{\lambda(t)}{t^{n-1}} = 0.
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There exists $c = c(p, n)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ such that

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and for every $w \in Q$ there exists arbitrarily small $r = r(w) > 0$ and a compact set $F = F(w, r)$ such that

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We first show how our result follows from this proposition.
Sketch of the Proof of Theorem A

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\( \mathcal{H}^{n-1}(P_m) < \infty \) for each positive integer \( m \) where

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P_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x, t))}{t^{n-1}} > \frac{1}{m} \right\}.
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Therefore,

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P = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x, t))}{t^{n-1}} > 0 \right\}
\]

\( \sigma \)-finite \( \mathcal{H}^{n-1} \) measure.

- Need to show: \( \mu(Q \setminus P) = 0 \).

- From Proposition 1 and measure theoretic arguments there exists a Borel set \( Q_1 \subset Q \) with

\[
\mu(\partial O \cap B(\hat{z}, \rho) \setminus Q_1) = 0 \text{ and } \mathcal{H}^{\lambda}(Q_1) = 0.
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$$P = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x, t))}{t^{n-1}} > 0 \right\}$$

is a $\sigma$-finite $\mathcal{H}^{n-1}$ measure.

Need to show: $\mu(Q \setminus P) = 0$.

From Proposition 1 and measure theoretic arguments there exists a Borel set $Q_1 \subset Q$ with

$$\mu(\partial O \cap B(\hat{z}, \rho) \setminus Q_1) = 0$$

and $\mathcal{H}^\lambda(Q_1) = 0$. 


\( \mathcal{H}^{n-1}(P_m) < \infty \) for each positive integer \( m \) where

\[
P_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x, t))}{t^{n-1}} > \frac{1}{m} \right\}.
\]

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\]
\[ \mu(Q \setminus \mathcal{P}) = 0. \]

Otherwise, there is a compact set \( K \subset Q \setminus \mathcal{P} \) and a positive non-decreasing \( \lambda_0 \) with
\[
\lim_{t \to 0} \frac{\lambda_0(t)}{t^{n-1}} = 0
\]
satisfying
\[
\mu(K) > 0 \quad \text{and} \quad \lim_{t \to 0} \frac{\mu(B(x, t))}{\lambda_0(t)} = 0 \quad \text{uniformly for } x \in K.
\]

This tells us that \( \mu \ll \mathcal{H}^{\lambda_0} \) on \( K \). Choose \( Q_1 \) relative to \( \lambda_0 \) to conclude that
\[
\mathcal{H}^{\lambda_0}(K \cap Q_1) = 0
\]
which will imply \( \mu(K \cap Q_1) = \mu(K) = 0 \).

\( \mu \) is concentrated on \( \mathcal{P} \) which has \( \sigma \)-finite \( \mathcal{H}^{n-1} \) measure. This finishes the proof of our result assuming Proposition 1.
\[ \mu(Q \setminus P) = 0. \]

Otherwise, there is a compact set \( K \subset Q \setminus P \) and a positive non-decreasing \( \lambda_0 \) with \( \lim_{t \to 0} \frac{\lambda_0(t)}{t^{n-1}} = 0 \) satisfying

\[ \mu(K) > 0 \text{ and } \lim_{t \to 0} \frac{\mu(B(x, t))}{\lambda_0(t)} = 0 \text{ uniformly for } x \in K. \]

This tells us that \( \mu \ll H^{\lambda_0} \) on \( K \). Choose \( Q_1 \) relative to \( \lambda_0 \) to conclude that \( H^{\lambda_0}(K \cap Q_1) = 0 \) which will imply \( \mu(K \cap Q_1) = \mu(K) = 0 \).}

\[ \mu \] is concentrated on \( P \) which has \( \sigma \)-finite \( H^{n-1} \) measure. This finishes the proof of our result assuming Proposition 1.
• $\mu(Q \setminus P) = 0$.

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• $\mu$ is concentrated on $P$ which has $\sigma$—finite $\mathcal{H}^{n-1}$ measure. This finishes the proof of our result assuming Proposition 1.
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• \( \mu \) is concentrated on \( P \) which has \( \sigma \)-finite \( \mathcal{H}^{n-1} \) measure. This finishes the proof of our result assuming Proposition 1.
Proposition 1

Let $\lambda$ be a non-decreasing function on $[0, 1]$ with

$$\lim_{t \to 0} \frac{\lambda(t)}{t^{n-1}} = 0.$$ 

There exists $c = c(p, n)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ such that

$$\mu(\partial O \cap B(\hat{z}, \rho) \setminus Q) = 0$$

and for every $w \in Q$ there exists arbitrarily small $r = r(w) > 0$ and a compact set $F = F(w, r)$ such that

$$\mathcal{H}^n(F) = 0 \text{ and } \frac{1}{c} \leq \mu(F).$$
Sketch of the Proof of Proposition 1

- Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when \( w = 0, \ B(0, 100) \subset B(\hat{z}, \rho) \).

- There is some \( c = c(p, n) \) and \( 2 \leq t \leq 50 \) such that

\[
\frac{1}{c} \leq \mu(B(0, 1)) \leq \max_{B(0, 2)} u \leq \max_{B(0, t)} u \leq c \mu(B(0, 100)) \leq c^2.
\]

To finish the proof of Proposition 1, it suffices to show for given small \( \epsilon, \tau > 0 \) that there exists a Borel set \( E \subset \partial O \cap B(0, 20) \) and \( c = c(p, n) \geq 1 \) with

\[
\mathcal{H}_{\tau}^{\lambda}(E) \leq \epsilon \text{ and } \mu(E) \geq \frac{1}{c}.
\]
Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when $w = 0$, $B(0, 100) \subset B(\hat{z}, \rho)$.

There is some $c = c(p, n)$ and $2 \leq t \leq 50$ such that

$$\frac{1}{c} \leq \mu(B(0, 1)) \leq \max_{B(0,2)} u \leq \max_{B(0,t)} u \leq c \mu(B(0, 100)) \leq c^2.$$ 

To finish the proof of Proposition 1, it suffices to show for given small $\epsilon, \tau > 0$ that there exists a Borel set $E \subset \partial O \cap B(0, 20)$ and $c = c(p, n) \geq 1$ with

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Sketch of the Proof of Proposition 1

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\]
A stopping time argument

Let $M$ a large positive number and $s < e^{-M}$. For each $z \in \partial O \cap B(0, 15)$ there is $t = t(z)$, $0 < t < 1$ with either

\[(\alpha) \quad \mu(B(z, t)) = Mt^{n-1}, t > s\]

or

\[(\beta) \quad t = s.\]

Use the Besicovitch covering theorem to get a covering $B(z_j, t_j)^N_1$ of $\partial O \cap B(0, 15)$ where $t_j = t(z_j)$ is the maximal for which either (\alpha) or (\beta) holds.
Let $M$ a large positive number and $s < e^{-M}$.
For each $z \in \partial O \cap B(0, 15)$ there is $t = t(z)$, $0 < t < 1$ with either

$$(\alpha) \ \mu(B(z, t)) = Mt^{n-1}, t > s$$

or

$$(\beta) \ t = s.$$
\[ \Omega := O \cap B(0, 15) \setminus \bigcup_{i=1}^{N} \overline{B}(z_i, t_i) \text{ and } D := \Omega \setminus \overline{B}(\tilde{z}, 2r_1) \]

\[ \Delta_p \hat{u} = 0 \]

\[ \hat{u} > 0 \]

- Let \( \hat{u} \) be the p-harmonic function in \( D \) with continuous boundary values, \( \hat{u} = \min_{\overline{B}(\tilde{z}, 2r_1)} u \) on \( \partial B(\tilde{z}, 2r_1) \) and \( \hat{u} = 0 \) on \( \partial \Omega \).

- Let \( \hat{\mu} \) be the p-harmonic measure associated with \( \hat{u} \).
Let \( \hat{u} \) be the \( p \)-harmonic function in \( D \) with continuous boundary values, \( \hat{u} = \min_{\overline{B}(\tilde{z}, 2r_1)} u \) on \( \partial \overline{B}(\tilde{z}, 2r_1) \) and \( \hat{u} = 0 \) on \( \partial \Omega \).

Let \( \hat{\mu} \) be the \( p \)-harmonic measure associated with \( \hat{u} \).
\[ \Omega := O \cap B(0, 15) \setminus \bigcup_{i=1}^{N} B(z_i, t_i) \] and \[ D := \Omega \setminus B(\tilde{z}, 2r_1) \]

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- Let \( \hat{\mu} \) be the p-harmonic measure associated with \( \hat{u} \).
• \( \hat{u} \leq u \) in \( D \).

• \( \partial \Omega \) is smooth except for a set of finite \( \mathcal{H}^{n-2} \).

• Using some barrier type estimate one can also show

\[
|\nabla \hat{u}| \leq cM^{\frac{1}{p-1}} \quad \text{in} \quad D.
\]

• Combining these we can show

\[
t_j^{1-n} \mu(\overline{B}(z_j, t_j)) \leq c t_j^{1-p} \max_{\overline{B}(z_j, 2t_j)} u^{p-1} \leq c^2 t_j^{1-n} \mu(B(z_j, 4t_j)).
\]
• $\hat{u} \leq u$ in $D$.

• $\partial \Omega$ is smooth except for a set of finite $\mathcal{H}^{n-2}$.

• Using some barrier type estimate one can also show

\[
|\nabla \hat{u}| \leq cM^{\frac{1}{p-1}} \text{ in } D.
\]

• Combining these we can show

\[
t_j^{1-n} \hat{\mu}(\overline{B}(z_j, t_j)) \leq c t_j^{1-p} \max_{B(z_j, 2t_j)} u^{p-1} \leq c^2 t_j^{1-n} \mu(B(z_j, 4t_j)).
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Combining these we can show

\[ t_j^{1-n} \hat{\mu}(\overline{B}(z_j, t_j)) \leq ct_j^{1-p} \max_{B(z_j, 2t_j)} u^{p-1} \leq c^2 t_j^{1-n} \mu(B(z_j, 4t_j)). \]
\[ \hat{u} \leq u \text{ in } D. \]

\[ \partial \Omega \text{ is smooth except for a set of finite } \mathcal{H}^{n-2}. \]

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Let $A >> 1$ be given

\{1, \ldots, N\} can be divided into disjoint subsets $\mathcal{G}, \mathcal{B},$ and $\mathcal{U}$ as

\[
\begin{align*}
\mathcal{G} &:= \{j : t_j > s\}, \\
\mathcal{B} &:= \{j : t_j = s \text{ and } |\nabla \hat{u}|^{p-1} \geq M^{-A} \text{ for some } x \in \partial \Omega \cap \partial B(z_j, t_j)\}, \\
\mathcal{U} &:= \{j : j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\}.
\end{align*}
\]

Define

\[E := \partial \Omega \cap \bigcup_{j \in \mathcal{G} \cup \mathcal{B}} B(z_j, t_j).\]

Easy to show $\mathcal{H}_T^\lambda (E) \leq \epsilon.$
Let \( A >> 1 \) be given

\( \{1, \ldots, N\} \) can be divided into disjoint subsets \( \mathcal{G}, \mathcal{B}, \) and \( \mathcal{U} \) as

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\begin{cases}
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\mathcal{U} := \{j : j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\}.
\end{cases}
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Define

\[
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\]

Easy to show \( \mathcal{H}^\lambda_f(E) \leq \epsilon \).
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$$

Define

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• Prove that

$$\int_{\partial \Omega} |\nabla \hat{u}|^{p-1} |\log |\nabla \hat{u}|| \, d\mathcal{H}^{n-1} \leq c' \log M.$$ 

• Use this to show

$$\hat{\mu}(\partial \Omega \cap \bigcup_{j \in U} \overline{B}(z_j, t_j)) \leq \hat{\mu}(\{x \in \partial \Omega : |\nabla \hat{u}(x)|^{p-1} \leq M^{-A}\}) \leq \frac{(p - 1)}{(A \log M)} \int_{\partial \Omega} |\nabla \hat{u}|^{p-1} |\log |\nabla \hat{u}|| \, d\mathcal{H}^{n-1} \leq \frac{c}{A}.$$ 

• $A$ is ours to choose, and choose it very large to make the measure of the $U$ set as small as we want.

• Use this to prove $\mu(E) \geq 1/c$. 
Prove that
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\[ \leq \frac{(p - 1)}{(A \log M)} \int_{\partial \Omega} |\nabla \hat{u}|^{p-1} |\log |\nabla \hat{u}|| \, d\mathcal{H}^{n-1} \leq \frac{c}{A}. \]

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• Use this to prove \( \mu(E) \geq 1/c. \)
There is an unpublished result of Jones-Wolff in [GM, Chapter IX, Theorem 3.1];

• **Jones-Wolff**: Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus \mathcal{C}$ where $\mathcal{C}$ is a “Cantor like” compact set. Then $\mathcal{H} - \text{dim } \omega < 1$.

Our aim is to generalize this result to $p$-harmonic measure, $\mu$, in $\mathbb{R}^n$ for $p \geq n \geq 2$ and for a certain domain.

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Part II: Example of a domain for which $\mathcal{H} - \dim \mu < n - 1$ for $p \geq n$.

There is an unpublished result of Jones-Wolff in [GM, Chapter IX, Theorem 3.1];

- **Jones-Wolff**: Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus \mathcal{C}$ where $\mathcal{C}$ is a "Cantor like" compact set. Then $\mathcal{H} - \dim \omega < 1$.

Our aim is to generalize this result to $p$-harmonic measure, $\mu$, in $\mathbb{R}^n$ for $p \geq n \geq 2$ and for a certain domain.

---

Construction of the domain

Let $S'$ be the square with side length $1/2$ and center $0$ in $\mathbb{R}^n$. $C_0 := S'$.

Let $Q_{11}, \ldots, Q_{14}$ be the squares of the four corners of $C_0$ of side length $a_1$, $0 < \alpha < a_1 < \beta < 1/4$, and let $C_1 = \bigcup_{i=1}^{4} Q_{1i}$.

Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the squares of corners of each $Q_{1i}$, $i = 1, \ldots, 4$ of side length $a_1a_2$, $\alpha < a_2 < \beta$. Let $C_2 = \bigcup_{j=1}^{16} Q_{2j}$.

By repeating the process we get $\mathcal{C}$. 

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$C_0$ $C_1$ $C_2$ $\mathcal{C}$
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Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the square of corners of each $Q_{1i}$, $i = 1, \ldots, 4$ of side length $a_1a_2$, $\alpha < a_2 < \beta$. Let $C_2 = \bigcup_{j=1}^{16} Q_{2j}$.

By repeating the process we get $C$. 

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Construction of the domain

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By repeating the process we get $C$. 

![Diagram showing the construction of the domain]
Let $\mathcal{S} = 2\mathcal{S}' \subset \mathbb{R}^n$ and let $u^\infty$ be a $p$-harmonic function in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u^\infty = 1$ on $\partial \mathcal{S}$ and $u^\infty = 0$ on $\mathcal{C}$. Let $\mu^\infty$ be the $p$-harmonic measure associated to $u^\infty$.

**Theorem B (A.-Lewis-Vogel)**

Then $\mathcal{H} - \dim \mu^\infty \leq n - 1 - \delta$ for some $\delta = \delta(p, n, \alpha, \beta) > 0$.

- Here $\delta \geq c^{-1}(p - n)$ where $c \geq 1$ can be chosen to depend only on $n, \alpha$, and $\beta$ when $p \in [n, n + 1]$. 
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Sketch of Proof of Theorem B

- Jones and Wolff used the idea of counting zeros of $\nabla G$.

- In higher dimensions and when $p \neq 2$ we then have a little control over the zeros of $\nabla u$.

Let

$$\tilde{\Gamma} = \{ \tilde{Q}_{k,j}; \ k = 1, \ldots, \text{ and } j = 1, \ldots, 2^{kn} \}.$$ 

Our result essentially follows from this Proposition;

**Proposition 2**

Let $\tilde{Q} \in \tilde{\Gamma}$ be a given cube. Then there exists $\delta' > 0$ with the same dependence as $\delta$ in Theorem B, $c = c(p, n, \alpha, \beta) \geq 1$, and a compact set $F \subset C \cap \tilde{Q}$ with

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In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution \( u \) with 0 boundary values for a larger class of quasilinear elliptic PDEs exists;

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\text{div} \cdot \mathcal{A}(x, \nabla u) = 0
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where \( \mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies certain structural assumptions. The measure is so called \( \mathcal{A} \)-harmonic measure.

In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

If \( \mathcal{A}(\xi) = |\xi|^{p-2}\xi \), then the above PDE becomes the usual \( p \)-Laplace equation.

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\{ \text{Laplace} \} \subset \{ \text{p-Laplace} \} \subset \{ \Delta_f u = 0 \} \subset \{ \mathcal{A} - \text{Harmonic PDEs} \}.
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Introduction to $\Delta f u = 0$

Let $p$ be fixed, $1 < p < \infty$. Let $f$ be a function with following properties;

1. $f : \mathbb{R}^n \to (0, \infty)$ is homogeneous of degree $p$. That is, $f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0$ when $\eta \in \mathbb{R}^n \setminus \{0\}$.

2. $Df = (f_{\eta_1}, \ldots, f_{\eta_n})$ has continuous partial derivatives when $\eta \neq 0$.

3. $f$ is uniformly convex in $B(0, 1) \setminus B(0, 1/2)$.

That is, There exists $c_* \geq 1$ such that for a.e. $\eta \in \mathbb{R}^n, 1/2 < |\eta| < 1$, and all $\xi \in \mathbb{R}^n$ we have $c_*^{-1} |\xi|^2 \leq \sum_{j,k=1}^n \frac{\partial^2 f}{\partial \eta_j \partial \eta_k}(\eta) \xi_j \xi_k \leq c_* |\xi|^2$. 
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We consider weak solutions, $u$, to the Euler Lagrange equation;

$$\Delta_f u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial f(\nabla u)}{\partial \eta_i} \right) = 0. \quad (1)$$

in $\Omega \cap N$ where $N$ is an open neighborhood of $\partial \Omega$. Assume also that $u > 0$ in $N \cap \Omega$ with continuous boundary values on $\partial \Omega$. Set $u \equiv 0$ in $N \setminus \Omega$ to have $u \in W^{1,p}(N)$ and $\Delta_f u = 0$ weakly in $N$. Then, there exists a unique finite positive Borel measure $\mu_f$ associated with $u$ having support contained in $\partial \Omega$ satisfying

$$\int \langle \nabla \eta f(\nabla u), \nabla \phi \rangle \, dx = - \int \phi \, d\mu_f \text{ whenever } \phi \in C_0^\infty(N).$$

- When $f(\eta) = |\eta|^2$ then (1) $\leadsto$ Laplace equation, $\Delta u = 0$.
- When $f(\eta) = |\eta|^p$, $1 < p < \infty$, then (1) $\leadsto$ p-Laplace equation, $\text{div}(|\nabla u|^{p-2}\nabla u) = 0$.
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Using this sub solution estimate and following arguments we have used for $p$ harmonic measure we show that

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*Theorem A and Theorem B hold for $\mu_f$.**
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More results in the plane for simply connected domains

Let \( n = 2 \) and \( \Omega \subset \mathbb{R}^2 \) be any bounded simply connected domain.

Let

\[
\lambda(r) := r \exp \left\{ A \sqrt{\log \frac{1}{r} \log \log \log \frac{1}{r}} \right\}.
\]

- **Makarov:** \( \omega \ll \mathcal{H}^\lambda \) if \( A \) is large enough.

For any small \( \epsilon > 0 \),

\[
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More results in the plane for simply connected domains

Let $n = 2$ and $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain.

Let

\[ \lambda(r) := r \exp \left\{ A \sqrt{\log \frac{1}{r} \log \log \log \frac{1}{r}} \right\}. \]

- **Makarov**: $\omega \ll \mathcal{H}^\lambda$ if $A$ is large enough.

For any small $\epsilon > 0$,

\[ \omega \ll \mathcal{H}^{1-\epsilon} \text{ and } \omega \perp \mathcal{H}^{1+\epsilon}. \]

Hence

\[ \mathcal{H} - \dim \omega = 1. \]
Let

\[ \hat{\lambda}(r) := r \exp \left\{ A \sqrt{\log \frac{1}{r} \log \log \frac{1}{r}} \right\}. \]

- **Lewis-Nyström-Poggi Corradini:**
  1. \( \mu_p \ll \mathcal{H}^{\hat{\lambda}} \) when \( 1 < p < 2 \) for some \( A = A(p) \geq 1 \).
  2. \( \mu_p \) is concentrated on a set of \( \sigma \)-finite \( \mathcal{H}^{\hat{\lambda}} \) when \( 2 < p < \infty \) for some \( A = A(p) \leq -1 \).

- **Lewis:**
  1. If \( 1 < p < 2 \), then \( \mu_p \ll \mathcal{H}^{\hat{\lambda}} \) for \( A = A(p) \) sufficiently large.
  2. If \( 2 < p < \infty \), then \( \mu_p \) is concentrated on a set of \( \sigma \)-finite \( \mathcal{H}^1 \).

\[ \mathcal{H} - \dim \mu_p \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ = 1 & \text{when } p = 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases} \]
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let $u > 0$ be a weak solution to $\triangle_f u = 0$ in $\Omega \cap N$. Let $\mu_f$ be the measure associated with $u$.

- Let $\hat{u}$ be a capacitary function for $D = \Omega \setminus B(z_0, d(z_0, \partial \Omega)/2)$ for some fixed $z_0 \in \Omega$.
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Then $\mu_f \ll \hat{\mu}_f \ll \mu_f$.

1. $\hat{u}_z$ is a $K$-quasiregular mapping.
2. $\hat{u}_z \neq 0$ in $D$.
3. $\hat{u}$ satisfies the so called fundamental inequality;
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4. $\zeta = \hat{u}$, $\zeta = \hat{u}_{x_k}$, $k = 1, 2$, are solution to $L \zeta = 0$. Moreover, $\log f(\nabla \hat{u})$ is a super solution when $1 < p < 2$, a solution when $p = 2$, and a sub solution when $2 < p < \infty$ to $L \zeta$ where
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• let \( u > 0 \) be a weak solution to \( \Delta_f u = 0 \) in \( \Omega \cap N \). Let \( \mu_f \) be the measure associated with \( u \).

• Let \( \hat{u} \) be a capacitary function for \( D = \Omega \setminus B(z_0, d(z_0, \partial \Omega)/2) \) for some fixed \( z_0 \in \Omega \).

**Capacitary function**

\( \hat{u} \) is called capacitary function for \( D \) if \( \hat{u} \) is positive weak solution to \( \Delta_f \hat{u} = 0 \) in \( D \) with continuous boundary values \( \hat{u} = 1 \) on \( \partial B(z_0, d(z_0, \partial \Omega)/2) \) and \( \hat{u} = 0 \) on \( \partial \Omega \).

• Let \( \hat{\mu}_f \) be the associated measure to \( \hat{u} \).

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let \( u > 0 \) be a weak solution to \( \Delta_f u = 0 \) in \( \Omega \cap N \). Let \( \mu_f \) be the measure associated with \( u \).

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\( \hat{u}_z \) is a \( K \)–quasiregular mapping.

**Quasiregular mapping**

\( \hat{u}_z \in W^{1,2} \) locally and \( |\hat{u}_z| \leq k|\hat{u}_z| \) a.e. in \( \Omega \), and \( k = (K - 1)/(K + 1) \) where

\[
\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)
\]

\( \hat{u}_z \neq 0 \) in \( D \).

\( \hat{u} \) satisfies the so called fundamental inequality;

\[
\hat{u}(z) \sim |\nabla \hat{u}(z)| \quad \text{for all} \quad z \text{ near} \ \partial \Omega.
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Endpoint-type results for $\mathcal{H} - \dim \mu_f$

A weaker version of Makarov’s and Lewis’ results;

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1. If $1 < p \leq 2$, there exists $A = A(p,f) \geq 1$ such that $\mu_f \ll \mathcal{H}^{\lambda}$.
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Theorem E (A., Work in Progress)

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When $\Omega$ is simply connected in the plane. Then

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Let $\Omega = C \cup \{\infty\} \setminus C$.

- **Batakis**: $\mathcal{H} - \text{dim } \omega \to 1$ when $\alpha \to 1/2$ in the plane.

Let $S$ be the unit cube centered at 0 in $\mathbb{R}^n$ and let $C$ be the four-corner Cantor set. Let $u$ be $p$-harmonic in $S \setminus C$ with continuous boundary values 1 on $\partial S$ and 0 on $C$. Let $\mu$ be the $p$-harmonic measure associated with $u$.

**Question 1**

$\alpha \to 1/2 \implies \mathcal{H} - \text{dim } \mu_p \to n - 1$ in $\mathbb{R}^n$?
Some Questions

Let $C$ be the *four-corner Cantor set* with $\{a_i\}$ where each $a_i$ satisfies $\alpha < a_i < \beta \leq 1/2$ in $\mathbb{R}^n$.

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On a conjecture of Øksendal

**Conjecture (Øksendal)**

\[ \mathcal{H} - \dim \omega < n - 1 \text{ for any domain } \Omega \subset \mathbb{R}^n. \]

This conjecture is false due to a result of Wolff; \( \mathcal{H} - \dim \omega > n - 1 \) for some snowflake domains.

Bourgain showed that \( \mathcal{H} - \dim \omega \leq n - \tau \) where \( \tau = \tau(n) \).

**Question 2**

What is the best value of \( \tau \)?

**Conjecture**

\[ \mathcal{H} - \dim \omega \leq n - 1 + \frac{n - 2}{n - 1}. \]

Fact: \( u \) is harmonic in the plane then \( \log |\nabla u| \) is subharmonic. (This fails in higher dimensions)

Substitute of the fact: if \( u \) is harmonic in \( \mathbb{R}^n \) then \( |\nabla u|^q \) is subharmonic for \( q \geq (n - 2)/(n - 1) \).
On a conjecture of Øksendal

Conjecture (Øksendal)

\[ \mathcal{H} - \dim \omega < n - 1 \text{ for any domain } \Omega \subset \mathbb{R}^n. \]

This conjecture is false due to a result of Wolff; \( \mathcal{H} - \dim \omega > n - 1 \) for some snowflake domains.

Bourgain showed that \( \mathcal{H} - \dim \omega \leq n - \tau \) where \( \tau = \tau(n) \).

Question 2

What is the best value of \( \tau \)?

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\[ \mathcal{H} - \dim \omega \leq n - 1 + \frac{n - 2}{n - 1}. \]

Fact: \( u \) is harmonic in the plane then \( \log |\nabla u| \) is subharmonic. (This fails in higher dimensions)

Substitute of the fact: if \( u \) is harmonic in \( \mathbb{R}^n \) then \( |\nabla u|^q \) is subharmonic for \( q \geq (n - 2)/(n - 1) \).
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THANKS!