Small perturbations of elliptic operators on 1-sided NTA domains satisfying the CDC

Murat Akman

Joint work with Steve Hofmann, José María Martell, and Tatiana Toro

Workshop on Real HA and its Applications to PDEs and GMT
On the occasion of the 60th birthday of Steve Hofmann
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Elliptic operators and elliptic measure

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain.
- L_0 be an elliptic operator associated to

$$L_0 u = -\operatorname{div}(A_0 \nabla u)$$

defined in the domain Ω with real, symmetric, and uniformly elliptic matrix $A_0 = A_0(X)$, i.e., there exists constant $\Lambda \geq 1$ satisfying

$$\Lambda^{-1}|\xi|^2 \leq \langle A_0(X)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

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- A domain Ω is called regular for the operator L_0 if for every $f \in C(\partial\Omega)$, there exists a (generalized) solution $u = u_f \in C(\bar{\Omega})$ which solves

$$\begin{cases} L_0 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

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- There exists a family of finite positive Borel probability measures $\{\omega_0^X\}_{X\in\Omega}$ whose support contained on $\partial\Omega$ satisfying

$$u(X) = \int_{\partial\Omega} f(z)\, d\omega_0^X(z).$$

Perturbation question

- Let A and A_0 be real, symmetric, and uniformly elliptic matrices s.t.

$$|A(x) - A_0(x)| = 0 \quad \text{whenever} \quad x \in \partial\Omega.$$

Consider

$$\text{“Good” operator} \begin{cases} L_0 u = -\operatorname{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

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Question
Suppose that we have “good estimates” for the Dirichlet problem for the operator L_0, under what optimal conditions, are those good estimates transferred to the Dirichlet problem for the operator L?

Good estimates: $\omega_0 \in A_\infty(\sigma)$, $\omega_0 \in \mathrm{RH}_p(\sigma)$ for some $1 < p < \infty$, $\omega_L \in \mathrm{RH}_2(\omega_0)$ etc...

Dark side of the linear elliptic operators

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- [Modica and Mortola '81] There exists elliptic operator L

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y}\left(\alpha \frac{\partial}{\partial y}\right) \quad \text{where} \quad \alpha \in C(\bar{\Omega}) \cap C^\infty(\Omega)$$

where $\partial\Omega \in C^\infty$ in the plane such that corresponding elliptic measure ω_L is not even absolutely continuous with respect to σ!!!

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It is not the domain Ω but the disagreement of A and A_0 that we should have conditions on...

Reverse Hölder class of q

- If a positive measure μ is absolutely continuous with respect to another positive measure ν on $\partial\Omega$ then there exists a function g such that $d\mu = g d\nu$. For $1 < q < \infty$, we say that the measure μ belongs to Reverse Hölder class of q with respect to ν on $\partial\Omega$, denoted by $\mu \in \mathrm{RH}_q(\nu, \partial\Omega) = \mathrm{RH}_q(\nu)$, if there exists a constant $C > 0$ such that

$$\left(\frac{1}{\nu(\Delta(x,r))} \int_{\Delta(x,r)} g^q d\nu \right)^{1/q} \leq C \frac{1}{\nu(\Delta(x,r))} \int_{\Delta(x,r)} g d\nu$$

for every surface ball $\Delta(x,r)$ centered on $\partial\Omega$ with $0 < r < \mathrm{diam}(\partial\Omega)$.

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- The union of the RH_q classes is the A_∞ class,

$$A_\infty(\nu) := \bigcup_{1<q<\infty} \mathrm{RH}_q(\nu).$$

In Juan's talk, μ is elliptic measure and ν is the surface measure σ.

Non-tangentially Accessible Domains(NTA)

- NTA $\equiv$ $\left\{ \begin{array}{l} \bullet\ \textbf{Interior}\text{ Corkscrew and Harnack Chain.} \\ \bullet\ \textbf{Exterior}\text{ Corkscrew.} \end{array} \right.$

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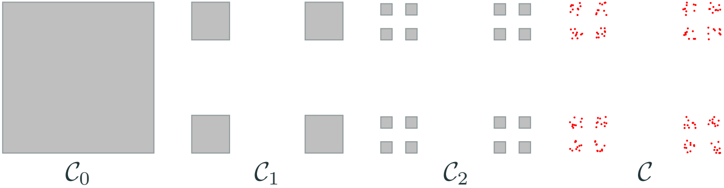
- 1-sided NTA $\equiv$ **Interior** Corkscrew and Harnack Chain.

Example: Complement of the classical four-corner Cantor set.



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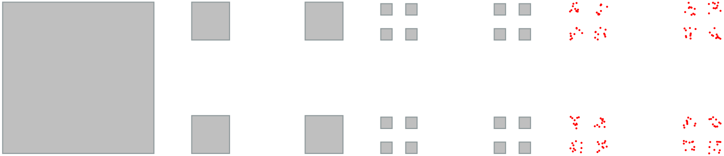

$\mathcal{C}_0$ $\mathcal{C}_1$ $\mathcal{C}_2$ $\mathcal{C}$

- $\partial\Omega$ is $n-$Ahlfors-David regular (ADR) if

$$cr^n \leq \mathcal{H}^n(\partial\Omega \cap B(z,r)) \leq cr^n$$

whenever $z \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$.

Carleson measure condition

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- In Juan's talk, we saw that the condition which preserves the good properties of the elliptic measures was the following Carleson measure condition for $\mathbf{a}(X)$

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} \frac{\mathbf{a}^2(X)}{\delta(X)} dX \right\}^{1/2} \leq C.$$

Here $\Delta = B(x, r) \cap \partial\Omega$ for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ and $T(\Delta) = B(x, r) \cap \Omega$.

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- Another "related" condition is smallness of

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} \mathbf{a}^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2}.$$

Here G_0 is the Green's function of Ω associated to the operator L_0 with pole at $0 \in \Omega$.

A relation

Let Ω be an NTA domain with Ahlfors regular boundary and $\omega_0 \in \mathsf{RH}_p(\sigma)$ for some $1 < p < \infty$.

- Given $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} \frac{\mathbf{a}^2(X)}{\delta(X)} dX \right\}^{1/2} \leq \delta$$

then

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} \mathbf{a}^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon.$$

Positive results

- [Fefferman, Kenig, and Pipher '91] Let Ω be a Lipschitz domain. There exists ϵ_0 depending on ellipticity constant and dimension such that if

$$(\star) \qquad \sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} \mathbf{a}^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0$$

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Aim: Study perturbation problem by replacing

1. NTA $\Rightarrow$ 1-sided NTA.
2. Ahlfors regularity $\Rightarrow$ "capacity density condition".

Capacity density condition

- Classical definition of Newtonian capacity of a compact set E in a domain D is
$$\mathsf{Cap}_{\Delta}(E,D) = \inf\left\{ \int |\nabla v|^2 dX : \, v \in C_0^{\infty}(D), \, v(x) \geq \mathbf{1}_E \right\}.$$

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- $\mathrm{Cap}(E,D)\approx \mathrm{Cap}_{\Delta}(E,D)$ due to ellipticity of A.
- Capacity density condition (CDC) if there exists a constant $c>0$ s.t.
$$\frac{\mathrm{Cap}(\bar{B}(w,r)\setminus\Omega, B(w,2r))}{\mathrm{Cap}(\bar{B}(w,r), B(w,2r))}\approx\frac{\mathrm{Cap}(\bar{B}(w,r)\setminus\Omega, B(w,2r))}{r^{n-1}}\geq c$$
for all $w\in\partial\Omega$ and $0<r<\operatorname{diam}(\partial\Omega)$.

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for all $w \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$.

- It is true that

$$\text{Ahlfors David regularity} \implies \text{CDC} \implies \text{Wiener regularity.}$$

First result - work in progress

Theorem (A., Hofmann, Martell, Toro)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. There exists $\epsilon_0 > 0$ depending on the ellipticity constant and dimension such that if

$$\sup_{\Delta \subseteq \partial\Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} \mathbf{a}^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0$$

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Proof: Uses ideas from [Cavero, Hofmann, and Martell '18] to change from A_0 to A "slowly" inside the domain and ideas from [Hofmann and Martell '12 and '14]; good properties of sawtooth domains, dyadic sawtooth lemma for projections.

Yet another condition

- Let $\mathcal{A}(\mathbf{a})(x) := \left(\iint_{\Gamma(x)} \frac{\mathbf{a}^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2}$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial\Omega$.

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- The connection is [if Ω is "nice"]

$$\frac{C}{\omega_0(\Delta)} \iint_{T(\Delta)} \mathbf{a}^2(X) \frac{G_0(X)}{\delta^2(X)} dX \leq \frac{1}{\omega_0(\Delta)} \int_{\Delta} (\mathcal{A}(\mathbf{a})(x))^2 d\omega_0(x).$$

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and $\omega_0 \in A_\infty(\sigma)$ then $\omega_L \in A_\infty(\sigma)$.

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Second result - work in progress

Theorem (A., Hofmann, Martell, Toro)
Let $\Omega \subset \mathbb{R}^{n+1}$ *be a 1-sided NTA domain with ADR boundary. If* $\|\mathcal{A}(\mathbf{a})\|_{L^\infty(\sigma)} \leq C_0 < \infty$ *where*

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Corollary
Let Ω be a uniform domain with Ahlfors regular boundary. Let ω_0 and ω_L be as above. If $\omega_L \in RH_p(\omega_0)$ for some $1 < p < \infty$ and $\omega_0 \in RH_q(\sigma)$ then $\omega_L \in RH_r(\sigma)$ where $r = \frac{pq}{p+q-1}$.

Thank you for your attention!!!

Happy birthday Steve...