Small perturbations of elliptic operators on 1-sided NTA domains satisfying the CDC

Murat Akman
Joint work with Steve Hofmann, José María Martell, and Tatiana Toro

Workshop on Real HA and its Applications to PDEs and GMT
On the occasion of the 60th birthday of Steve Hofmann
ICMAT, Madrid (Spain), May 28 - June 1, 2018
Elliptic operators and elliptic measure

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain.

- $L_0$ be an elliptic operator associated to

$$L_0 u = - \text{div}(A_0 \nabla u)$$

defined in the domain $\Omega$ with real, symmetric, and uniformly elliptic matrix $A_0 = A_0(X)$, i.e., there exists constant $\Lambda \geq 1$ satisfying

$$\Lambda^{-1} |\xi|^2 \leq \langle A_0(X)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$
Let \( \Omega \subset \mathbb{R}^{n+1}, n \geq 2 \), be a bounded domain.

\( L_0 \) be an elliptic operator associated to

\[
L_0 u = - \text{div}(A_0 \nabla u)
\]

defined in the domain \( \Omega \) with real, symmetric, and uniformly elliptic matrix \( A_0 = A_0(X) \), i.e., there exists constant \( \Lambda \geq 1 \) satisfying

\[
\Lambda^{-1} |\xi|^2 \leq \langle A_0(X)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall X \in \Omega \text{ and } \xi \in \mathbb{R}^n.
\]

A domain \( \Omega \) is called regular for the operator \( L_0 \) if for every \( f \in C(\partial \Omega) \), there exists a (generalized) solution \( u = u_f \in C(\overline{\Omega}) \) which solves

\[
\begin{cases}
L_0 u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]
Elliptic operators and elliptic measure

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded domain.

$L_0$ be an elliptic operator associated to

$$L_0 u = -\text{div}(A_0 \nabla u)$$

defined in the domain $\Omega$ with real, symmetric, and uniformly elliptic matrix $A_0 = A_0(X)$, i.e., there exists constant $\Lambda \geq 1$ satisfying

$$\Lambda^{-1} |\xi|^2 \leq \langle A_0(X)\xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall X \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

A domain $\Omega$ is called regular for the operator $L_0$ if for every $f \in C'(\partial \Omega)$, there exists a (generalized) solution $u = u_f \in C(\overline{\Omega})$ which solves

$$\begin{cases} 
L_0 u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}$$

There exists a family of finite positive Borel probability measures $\{\omega_0^X\}_{X \in \Omega}$ whose support contained on $\partial \Omega$ satisfying

$$u(X) = \int_{\partial \Omega} f(z) d\omega_0^X(z).$$
Perturbation question

Let $A$ and $A_0$ be real, symmetric, and uniformly elliptic matrices s.t.

$$|A(x) - A_0(x)| = 0 \quad \text{whenever} \quad x \in \partial \Omega.$$ 

Consider

"Good" operator \[
\begin{cases} 
L_0 u = - \text{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega
\end{cases}
\]

Perturbed operator \[
\begin{cases} 
Lu = - \text{div}(A \nabla u) = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega
\end{cases}
\]

Let $\omega_0$ and $\omega_L$ be the elliptic measures of $\Omega$ associate to $L_0$ and $L$ respectively with pole at $0 \in \Omega$. 

Suppose that we have "good estimates" for the Dirichlet problem for the operator $L_0$, under what optimal conditions, are those good estimates transferred to the Dirichlet problem for the operator $L$?
Perturbation question

Let \( A \) and \( A_0 \) be real, symmetric, and uniformly elliptic matrices s.t.
\[
|A(x) - A_0(x)| = 0 \quad \text{whenever} \quad x \in \partial \Omega.
\]

Consider

\[
\text{“Good” operator } \begin{cases}
    L_0 u = -\text{div}(A_0 \nabla u) = 0 & \text{in } \Omega, \\
    u = f & \text{on } \partial \Omega
\end{cases}
\]

\[
\text{Perturbed operator } \begin{cases}
    L u = -\text{div}(A \nabla u) = 0 & \text{in } \Omega, \\
    u = f & \text{on } \partial \Omega
\end{cases}
\]

Let \( \omega_0 \) and \( \omega_L \) be the elliptic measures of \( \Omega \) associate to \( L_0 \) and \( L \) respectively with pole at \( 0 \in \Omega \).

**Question**

Suppose that we have “good estimates” for the Dirichlet problem for the operator \( L_0 \), under what optimal conditions, are those good estimates transferred to the Dirichlet problem for the operator \( L \)?

Good estimates: \( \omega_0 \in A_{\infty}(\sigma) \), \( \omega_0 \in \text{RH}_p(\sigma) \) for some \( 1 < p < \infty \), \( \omega_L \in \text{RH}_2(\omega_0) \) etc...
Dark side of the linear elliptic operators

- [Littman, Stampacchia, and Weinberger '63] \( \Omega \) is regular for \( L_0 = \Delta \) iff it is regular for \( L \).

- [Modica and Mortola '81] There exists an elliptic operator \( L \) where \( \alpha \in C(\bar{\Omega}) \cap C^\infty(\Omega) \) and \( \partial \Omega \in C^\infty \) in the plane such that the corresponding elliptic measure \( \omega_L \) is not even absolutely continuous with respect to \( \sigma \).

- [Caffarelli, Fabes, and Kenig '81] There exists an elliptic operator \( L \) whose coefficient matrix is continuous on the closure of the unit disc of \( \mathbb{R}^2 \) but the corresponding elliptic measure is completely singular with respect to arclength on the unit circle.
Dark side of the linear elliptic operators

- [Littman, Stampacchia, and Weinberger '63] \( \Omega \) is regular for \( L_0 = \Delta \) iff it is regular for \( L \).

- Easy fact: If \( \Omega = B(0, 1) \subset \mathbb{R}^{n+1} \) then harmonic measure \( \omega^0 \) is the normalized surface measure \( \sigma := \mathcal{H}^n|_{\partial \Omega} \).
[Littman, Stampacchia, and Weinberger '63] Ω is regular for $L_0 = \Delta$ iff it is regular for $L$.

Easy fact: If $\Omega = B(0, 1) \subset \mathbb{R}^{n+1}$ then harmonic measure $\omega^0$ is the normalized surface measure $\sigma := \mathcal{H}^n|_{\partial \Omega}$.

[Modica and Mortola '81] There exists elliptic operator $L$

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left( \alpha \frac{\partial}{\partial y} \right)$$

where $\alpha \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ where $\partial \Omega \in C^\infty$ in the plane such that corresponding elliptic measure $\omega_L$ is not even absolutely continuous with respect to $\sigma$!!!

[Caffarelli, Fabes, and Kenig '81] There exists an elliptic operator $L$ whose coefficient matrix is continuous on the closure of the unit disc of $\mathbb{R}^2$ but the corresponding elliptic measure is completely singular with respect to arclength on the unit circle!!!
[Littman, Stampacchia, and Weinberger '63] \( \Omega \) is regular for \( L_0 = \Delta \) iff it is regular for \( L \).

Easy fact: If \( \Omega = B(0, 1) \subset \mathbb{R}^{n+1} \) then harmonic measure \( \omega^0 \) is the normalized surface measure \( \sigma := \mathcal{H}^n|_{\partial \Omega} \).

[Modica and Mortola '81] There exists elliptic operator \( L \)

\[
L = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} \left( \alpha \frac{\partial}{\partial y} \right)
\]

where \( \alpha \in C(\overline{\Omega}) \cap C^\infty(\Omega) \) where \( \partial \Omega \in C^\infty \) in the plane such that corresponding elliptic measure \( \omega_L \) is not even absolutely continuous with respect to \( \sigma \)!!!

[Caffarelli, Fabes, and Kenig '81] There exists an elliptic operator \( L \) whose coefficient matrix is continuous on the closure of the unit disc of \( \mathbb{R}^2 \) but the corresponding elliptic measure is completely singular with respect to arclength on the unit circle!!!

It is not the domain \( \Omega \) but the disagreement of \( A \) and \( A_0 \) that we should have conditions on...
Reverse Hölder class of $q$

If a positive measure $\mu$ is absolutely continuous with respect to
another positive measure $\nu$ on $\partial \Omega$ then there exists a function $g$ such
that $d\mu = gd\nu$. For $1 < q < \infty$, we say that the measure $\mu$ belongs to
Reverse Hölder class of $q$ with respect to $\nu$ on $\partial \Omega$, denoted by
$\mu \in RH_q(\nu, \partial \Omega) = RH_q(\nu)$, if there exists a constant $C > 0$ such that

$$\left( \frac{1}{\nu(\Delta(x, r))} \int_{\Delta(x, r)} g^q d\nu \right)^{1/q} \leq C \frac{1}{\nu(\Delta(x, r))} \int_{\Delta(x, r)} gd\nu$$

for every surface ball $\Delta(x, r)$ centered on $\partial \Omega$ with $0 < r < \text{diam}(\partial \Omega)$. 

The union of the RH$_q$ classes is the A$_\infty$ class,
$A_\infty(\nu) := \bigcup_{1 < q < \infty} RH_q(\nu)$. 

In Juan's talk, $\mu$ is elliptic measure and $\nu$ is the surface measure $\sigma$. 
Reverse Hölder class of $q$

- If a positive measure $\mu$ is absolutely continuous with respect to another positive measure $\nu$ on $\partial \Omega$ then there exists a function $g$ such that $d\mu = gd\nu$. For $1 < q < \infty$, we say that the measure $\mu$ belongs to Reverse Hölder class of $q$ with respect to $\nu$ on $\partial \Omega$, denoted by $\mu \in RH_q(\nu, \partial \Omega) = RH_q(\nu)$, if there exists a constant $C > 0$ such that

$$\left( \frac{1}{\nu(\Delta(x, r))} \int_{\Delta(x, r)} g^q d\nu \right)^{1/q} \leq C \frac{1}{\nu(\Delta(x, r))} \int_{\Delta(x, r)} gd\nu$$

for every surface ball $\Delta(x, r)$ centered on $\partial \Omega$ with $0 < r < \text{diam}(\partial \Omega)$.

- The union of the $RH_q$ classes is the $A_\infty$ class,

$$A_\infty(\nu) : = \bigcup_{1<q<\infty} RH_q(\nu).$$

In Juan’s talk, $\mu$ is elliptic measure and $\nu$ is the surface measure $\sigma$. 
Non-tangentially Accessible Domains (NTA)

\[ \text{NTA} \equiv \begin{cases} \text{Interior} & \text{Corkscrew and Harnack Chain.} \\ \text{Exterior} & \text{Corkscrew.} \end{cases} \]

Example: Complement of the classical four-corner Cantor set.

\[ C_0 \subset C_1 \subset C_2 \]

\[ \partial \Omega \text{ is \ } n-\text{Ahlfors-David regular (ADR) if } cr_n \leq H_n(\partial \Omega \cap B(z, r)) \leq cr_n \text{ whenever } z \in \partial \Omega \text{ and } 0 < r < \text{diam}(\partial \Omega). \]
Non-tangentially Accessible Domains (NTA)

- **NTA** \(\equiv\) \(\bigg\{\)
  - **Interior** Corkscrew and Harnack Chain.
  - **Exterior** Corkscrew.

- 1-sided NTA \(\equiv\) **Interior** Corkscrew and Harnack Chain.

Example: Complement of the classical four-corner Cantor set.

\[
\begin{align*}
\partial \Omega & \text{ is } n^{-\text{Ahlfors-David regular (ADR)}} \text{ if } \\
& \text{whenever } z \in \partial \Omega \text{ and } 0 < r < \text{diam} (\partial \Omega).
\end{align*}
\]
Non-tangentially Accessible Domains (NTA)

\[ \text{NTA} \equiv \begin{cases} \text{Interior} \text{ Corkscrew and Harnack Chain.} \\ \text{Exterior} \text{ Corkscrew.} \end{cases} \]

\[ \text{1-sided NTA} \equiv \text{Interior} \text{ Corkscrew and Harnack Chain.} \]

Example: Complement of the classical four-corner Cantor set.

\[ \partial \Omega \text{ is } n-\text{Ahlfors-David regular (ADR)} \text{ if} \]

\[ cr^n \leq \mathcal{H}^n(\partial \Omega \cap B(z, r)) \leq cr^n \]

whenever \( z \in \partial \Omega \) and \( 0 < r < \text{diam}(\partial \Omega) \).
Let

\[ a(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|. \]
Let

\[ a(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|. \]

In Juan’s talk, we saw that the condition which preserves the good properties of the elliptic measures was the following Carleson measure condition for \( a(X) \)

\[ \sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq C. \]

Here \( \Delta = B(x, r) \cap \partial \Omega \) for \( x \in \partial \Omega \) and \( 0 < r < \text{diam}(\partial \Omega) \) and \( T(\Delta) = B(x, r) \cap \Omega. \)
Carleson measure condition

Let
\[
a(X) := \sup_{Y \in B(X, \delta(X)/2)} |A(Y) - A_0(Y)|.
\]

In Juan’s talk, we saw that the condition which preserves the good properties of the elliptic measures was the following Carleson measure condition for \(a(X)\)

\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\sigma(\Delta)} \int \int_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq C.
\]

Here \(\Delta = B(x, r) \cap \partial \Omega\) for \(x \in \partial \Omega\) and \(0 < r < \text{diam}(\partial \Omega)\) and \(T(\Delta) = B(x, r) \cap \Omega\).

Another “related” condition is smallness of

\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2}.
\]

Here \(G_0\) is the Green’s function of \(\Omega\) associated to the operator \(L_0\) with pole at \(0 \in \Omega\).
Let \( \Omega \) be an NTA domain with Ahlfors regular boundary and \( \omega_0 \in RH_p(\sigma) \) for some \( 1 < p < \infty \).

Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if
\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\sigma(\Delta)} \iint_{T(\Delta)} \frac{a^2(X)}{\delta(X)} dX \right\}^{1/2} \leq \delta
\]
then
\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon.
\]
Positive results

[Fefferman, Kenig, and Pipher ’91] Let $\Omega$ be a Lipschitz domain. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if

\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0
\]

then $\omega_L \in RH_2(\omega_0)$. 

[Milakis, Pipher, and Toro ’13] Let $\Omega$ be an NTA domain with ADR boundary. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if (*) holds then $\omega_L \in RH_2(\omega_0)$. 

Question: Can we go beyond NTA domains? Can we relax Ahlfors regularity of the boundary?

Aim: Study perturbation problem by replacing

1. NTA $\Rightarrow$ 1-sided NTA.
2. Ahlfors regularity $\Rightarrow$ "capacity density condition".
Positive results

[Fefferman, Kenig, and Pipher ’91] Let \( \Omega \) be a Lipschitz domain. There exists \( \epsilon_0 \) depending on ellipticity constant and dimension such that if

\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \iint_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} \, dX \right\}^{1/2} \leq \epsilon_0
\]

then \( \omega_L \in \text{RH}_2(\omega_0) \).

[Milakis, Pipher, and Toro ’13] Let \( \Omega \) be an NTA domain with ADR boundary. There exists \( \epsilon_0 \) depending on ellipticity constant and dimension such that if \((\star)\) holds then \( \omega_L \in \text{RH}_2(\omega_0) \).
Positive results

- [Fefferman, Kenig, and Pipher ’91] Let $\Omega$ be a Lipschitz domain. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if

$$(★) \quad \sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0$$

then $\omega_L \in RH_2(\omega_0)$.

- [Milakis, Pipher, and Toro ’13] Let $\Omega$ be an NTA domain with ADR boundary. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if $(★)$ holds then $\omega_L \in RH_2(\omega_0)$.

**Question**

*Can we go beyond NTA domains? Can we relax Ahlfors regularity of the boundary?*
Positive results

[Fefferman, Kenig, and Pipher '91] Let $\Omega$ be a Lipschitz domain. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if

\[
\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0
\]

then $\omega_L \in RH_2(\omega_0)$.

[Milakis, Pipher, and Toro '13] Let $\Omega$ be an NTA domain with ADR boundary. There exists $\epsilon_0$ depending on ellipticity constant and dimension such that if $(\star)$ holds then $\omega_L \in RH_2(\omega_0)$.

**Question**

*Can we go beyond NTA domains? Can we relax Ahlfors regularity of the boundary?*

**Aim:** Study perturbation problem by replacing

1. NTA $\Rightarrow$ 1-sided NTA.
2. Ahlfors regularity $\Rightarrow$ “capacity density condition”.
Capacity density condition

- Classical definition of Newtonian capacity of a compact set $E$ in a domain $D$ is

  $$\text{Cap}_\Delta(E, D) = \inf \left\{ \int |\nabla v|^2 dX : v \in C_0^\infty(D), \ v(x) \geq 1_E \right\}.$$
Capacity density condition

- Classical definition of Newtonian capacity of a compact set $E$ in a domain $D$ is

$$\text{Cap}_\Delta(E, D) = \inf \left\{ \int |\nabla v|^2 dX : v \in C^\infty_0(D), v(x) \geq 1_E \right\}.$$ 

- We define $L$-Capacity associated to $Lu = -\text{div}(A\nabla u)$ by

$$\text{Cap}(E, D) = \inf \left\{ \int \langle A\nabla v, \nabla v \rangle dX ; v \in C^\infty_0(D), v(x) \geq 1_E \right\}.$$ 

- $\text{Cap}(E, D) \approx \text{Cap}_\Delta(E, D)$ due to ellipticity of $A$. 

- Capacity density condition (CDC) if there exists a constant $c > 0$ s.t.

$$\text{Cap}(\overline{B}(w, r) \setminus \Omega, B(w, 2r)) \approx \text{Cap}(\overline{B}(w, r) \setminus \Omega, B(w, 2r)) \approx \text{Cap}(\overline{B}(w, r) \setminus \Omega, B(w, 2r)) \geq c \quad \text{for all } w \in \partial \Omega \text{ and } 0 < r < \text{diam}(\partial \Omega).$$ 

- It is true that Ahlfors David regularity $\Rightarrow$ CDC $\Rightarrow$ Wiener regularity.
Capacity density condition

- Classical definition of Newtonian capacity of a compact set $E$ in a domain $D$ is

$$\text{Cap}_\Delta(E, D) = \inf \left\{ \int |\nabla v|^2 dX : v \in C_0^\infty(D), v(x) \geq 1_E \right\}.$$ 

- We define $L$-Capacity associated to $Lu = -\text{div}(A \nabla u)$ by

$$\text{Cap}(E, D) = \inf \left\{ \int \langle A \nabla v, \nabla v \rangle dX ; v \in C_0^\infty(D), v(x) \geq 1_E \right\}.$$ 

- $\text{Cap}(E, D) \approx \text{Cap}_\Delta(E, D)$ due to ellipticity of $A$.

- Capacity density condition (CDC) if there exists a constant $c > 0$ s.t.

$$\frac{\text{Cap}(\bar{B}(w, r) \setminus \Omega, B(w, 2r))}{\text{Cap}(ar{B}(w, r), B(w, 2r))} \approx \frac{\text{Cap}(\bar{B}(w, r) \setminus \Omega, B(w, 2r))}{r^{n-1}} \geq c$$

for all $w \in \partial \Omega$ and $0 < r < \text{diam}(\partial \Omega)$. 

Capacity density condition

- Classical definition of Newtonian capacity of a compact set $E$ in a domain $D$ is
  \[
  \text{Cap}_{\Delta}(E, D) = \inf \left\{ \int |\nabla v|^2 dX : v \in C_0^\infty(D), v(x) \geq 1_E \right\}.
  \]

- We define $L$-Capacity associated to $Lu = -\text{div}(A\nabla u)$ by
  \[
  \text{Cap}(E, D) = \inf \left\{ \int \langle A\nabla v, \nabla v \rangle dX ; v \in C_0^\infty(D), v(x) \geq 1_E \right\}.
  \]

- $\text{Cap}(E, D) \approx \text{Cap}_{\Delta}(E, D)$ due to ellipticity of $A$.

- Capacity density condition (CDC) if there exists a constant $c > 0$ s.t.
  \[
  \frac{\text{Cap}(\bar{B}(w, r) \setminus \Omega, B(w, 2r))}{\text{Cap}(\bar{B}(w, r), B(w, 2r))} \approx \frac{\text{Cap}(\bar{B}(w, r) \setminus \Omega, B(w, 2r))}{r^{n-1}} \geq c
  \]
  for all $w \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$.

- It is true that
  \[
  \text{Ahlfors David regularity} \implies \text{CDC} \implies \text{Wiener regularity}.
  \]
Theorem (A., Hofmann, Martell, Toro)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. There exists $\epsilon_0 > 0$ depending on the ellipticity constant and dimension such that if

$$\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int\int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0$$

then $\omega_L \in RH_2(\omega_0)$.

▶ i.e., the Dirichlet problem for $L_1$ is solvable with data in $L^2(d\omega_0)$. 
Theorem (A., Hofmann, Martell, Toro)

Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain satisfying the CDC. There exists $\epsilon_0 > 0$ depending on the ellipticity constant and dimension such that if

$$\sup_{\Delta \subseteq \partial \Omega} \left\{ \frac{1}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \right\}^{1/2} \leq \epsilon_0$$

then $\omega_L \in RH_2(\omega_0)$.

\[ \downarrow \text{i.e., the Dirichlet problem for } L_1 \text{ is solvable with data in } L^2(d\omega_0). \]

Proof: Uses ideas from [Cavero, Hofmann, and Martell '18] to change from $A_0$ to $A$ “slowly” inside the domain and ideas from [Hofmann and Martell '12 and '14]; good properties of sawtooth domains, dyadic sawtooth lemma for projections.
Yet another condition

Let \( A(a)(x) := \left( \int \int_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2} \)

where \( \Gamma(x) \) is the non-tangential cone with vertex at \( x \in \partial \Omega \).
Yet another condition

Let \( A(a)(x) := \left( \iint_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2} \)

where \( \Gamma(x) \) is the non-tangential cone with vertex at \( x \in \partial \Omega \).

The connection is [if \( \Omega \) is “nice”]

\[
\frac{C}{\omega_0(\Delta)} \iint_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \leq \frac{1}{\omega_0(\Delta)} \int_{\Delta} (A(a)(x))^2 d\omega_0(x).
\]
Let \( A(a)(x) := \left( \int \int_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} \, dX \right)^{1/2} \)

where \( \Gamma(x) \) is the non-tangential cone with vertex at \( x \in \partial \Omega \).

The connection is [if \( \Omega \) is “nice”]

\[
\frac{C}{\omega_0(\Delta)} \int \int_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} \, dX \leq \frac{1}{\omega_0(\Delta)} \int_{\Delta} (A(a)(x))^2 \, d\omega_0(x).
\]

[Fefferman '89 and Fefferman, Kenig, and Pipher '91] Let \( \Omega \) be a Lipschitz domain. If

\[ (\star\star) \quad \|A(a)\|_{L^\infty(\sigma)} \leq C_0 < \infty \]

and \( \omega_0 \in A_\infty(\sigma) \) then \( \omega_L \in A_\infty(\sigma) \).

This is sharp in the sense that \( \|A(a)\|_{L^\infty(\sigma)} \leq C_0 \) does not imply \( \omega_L \in RH_p(\sigma) \) if \( \omega_0 \in RH_p(\sigma) \).
Yet another condition

Let $A(a)(x) := \left( \iint_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2}$

where $\Gamma(x)$ is the non-tangential cone with vertex at $x \in \partial \Omega$.

The connection is [if $\Omega$ is “nice”]

$$\frac{C}{\omega_0(\Delta)} \iint_{T(\Delta)} a^2(X) \frac{G_0(X)}{\delta^2(X)} dX \leq \frac{1}{\omega_0(\Delta)} \int_{\Delta} (A(a)(x))^2 d\omega_0(x).$$

[Fefferman '89 and Fefferman, Kenig, and Pipher '91] Let $\Omega$ be a Lipschitz domain. If

$$\|A(a)\|_{L^\infty(\sigma)} \leq C_0 < \infty$$

and $\omega_0 \in A_\infty(\sigma)$ then $\omega_L \in A_\infty(\sigma)$.

This is sharp in the sense that $\|A(a)\|_{L^\infty(\sigma)} \leq C_0$ does not imply $\omega_L \in RH_p(\sigma)$ if $\omega_0 \in RH_p(\sigma)$.

[Milakis, Pipher, and Toro '13] Let $\Omega$ be a NTA domain with ADR boundary. If (**) holds and $\omega_0 \in A_\infty(\sigma)$ then $\omega_L \in A_\infty(\sigma)$. 
Theorem (A., Hofmann, Martell, Toro)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain with ADR boundary. If
\[ \|A(a)\|_{L^{\infty}(\sigma)} \leq C_0 < \infty \]
where
\[ A(a)(x) = \left( \iiint_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2} \]
and $\omega_0 \in A_{\infty}(\sigma)$ then $\omega_L \in A_{\infty}(\sigma)$. 
Theorem (A., Hofmann, Martell, Toro)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided NTA domain with ADR boundary. If
\[ \|A(a)\|_{L^\infty(\sigma)} \leq C_0 < \infty \]
where
\[ A(a)(x) = \left( \iint_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2} \]
and $\omega_0 \in A_\infty(\sigma)$ then $\omega_L \in A_\infty(\sigma)$.

Proof: Use Fubini’s Theorem and study elliptic measure associated to the operator $(1 - t)A_0 + tA$. Repeated application of our first result and transitivity of $A_\infty$ property will complete the proof.
Theorem (A., Hofmann, Martell, Toro)
Let \( \Omega \subset \mathbb{R}^{n+1} \) be a 1-sided NTA domain with ADR boundary. If \( \|A(a)\|_{L^{\infty}(\sigma)} \leq C_0 < \infty \) where
\[
A(a)(x) = \left( \int \int_{\Gamma(x)} \frac{a^2(X)}{\delta^{n+1}(X)} dX \right)^{1/2}
\]
and \( \omega_0 \in A_{\infty}(\sigma) \) then \( \omega_L \in A_{\infty}(\sigma) \).

Proof: Use Fubini’s Theorem and study elliptic measure associated to the operator \((1 - t)A_0 + tA\). Repeated application of our first result and transitivity of \( A_{\infty} \) property will complete the proof.

Corollary
Let \( \Omega \) be a uniform domain with Ahlfors regular boundary. Let \( \omega_0 \) and \( \omega_L \) be as above. If \( \omega_L \in RH_p(\omega_0) \) for some \( 1 < p < \infty \) and \( \omega_0 \in RH_q(\sigma) \) then \( \omega_L \in RH_r(\sigma) \) where \( r = \frac{pq}{p+q-1} \).
Thank you for your attention!!!

Happy birthday Steve...