The Brunn-Minkowski Inequality and a Minkowski Problem for Nonlinear Capacities

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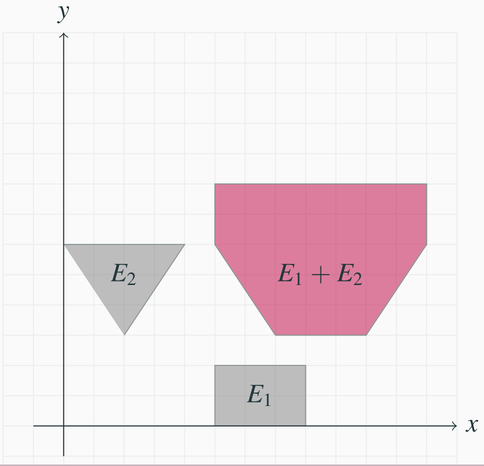
Minkowski Addition of Sets

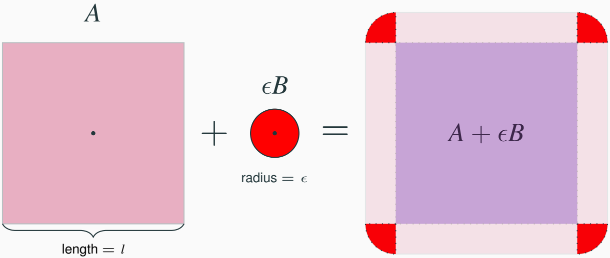
Let E_1 and E_2 be convex bodies (compact convex sets with non-empty interiors). Minkowski addition of $E_1 + E_2$ is defined as

$$E_1 + E_2 := \{a + b \,|\, a \in E_1,\, b \in E_2\} = \bigcup_{b \in E_2} E_1 + \{b\}.$$



$$sA := \{sa \,|\, a \in A\}.$$

Minkowski sum of a square and a disk



Therefore,

$$\begin{aligned} |A + \epsilon B| &= |A| + 4l\epsilon + |\epsilon B| \\ &\geq |A| + 2\sqrt{\pi} l\epsilon + |\epsilon B| \\ &= |A| + 2\sqrt{|A||\epsilon B|} + |\epsilon B|. \end{aligned}$$

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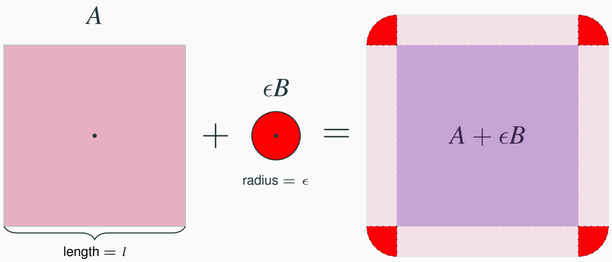

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Hence $\quad |A + \epsilon B|^{1/2} \geq |A|^{1/2} + |\epsilon B|^{1/2}.$

Of course this is not a coincidence!

The Brunn-Minkowski inequality

Theorem (Minkowski in $\sim$1887 and Brunn in $\sim$1896)
Let A and B two convex sets in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then

$$|(1-\lambda)A + \lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}}.$$

Moreover, equality holds iff A is a homothetic copy of B.*

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*: i.e. are equal up to translation and dilation.

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- The function $|\cdot|^{1/n}$ is a concave function in the class of convex sets in $\mathbb{R}^n$ under the Minkowski addition.
- It also holds for bounded measurable sets in $\mathbb{R}^n$. [Due to Lusternik in 1935, Hadwiger and Ohmann in 1956.]
- The Brunn-Minkowski Inequality implies the isoperimetric inequalities.
- Connections with: Sobolev inequality, Poincaré inequality, Young inequality, Prékopa-Leindler inequality, etc.

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Equivalent forms of the Brunn-Minkowski inequality

Let A, B be convex bodies in $\mathbb{R}^n$ and let $\lambda \in [0, 1]$. Then TFAE.

- Classic
$$|(1-\lambda)A + \lambda B|^{\frac{1}{n}} \geq (1-\lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}}.$$
- Elegant
$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$
- Multiplicative
$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^{\lambda}.$$
- Minimal
$$|(1-\lambda)A + \lambda B| \geq \min\{|A|, |B|\}.$$

Is it special to Lebesgue measure?

Let $\mathcal{K} = \{\text{Convex bodies in } \mathbb{R}^n\}$.

- $|\cdot| \geq 0$ is a homogeneous of degree n;

$$|tK| = t^n |K| \quad \text{whenever} \quad t \geq 0 \text{ and } K \in \mathcal{K}.$$

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Let $\mathbf{F} : \mathcal{K} \to \mathbb{R}$ be a functional such that

- $\mathbf{F}(K) \geq 0$ whenever $K \in \mathcal{K}$.
- $\mathbf{F}$ is α-homogeneous ($\alpha \neq 0$);

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Is there any other homogeneous of degree α functional $\mathbf{F}$ which verifies the Brunn-Minkowski inequality for convex bodies?

$$\left[\mathbf{F}((1-\lambda)A + \lambda B)\right]^{\frac{1}{\alpha}} \geq (1-\lambda)\left[\mathbf{F}(A)\right]^{\frac{1}{\alpha}} + \lambda\left[\mathbf{F}(B)\right]^{\frac{1}{\alpha}}.$$

What happens in the case of equality?

Inequalities of Brunn-Minkowski type

- Torsional rigidity. $\rightarrow \alpha = n + 2$. (Borell in 1985, Colesanti in 2005).
- Eigenvalue of the Monge-Ampère equation. $\rightarrow \alpha = -2n$. (Salani in 2005).
- Homogeneous Minkowski-concave function of degree m. $\rightarrow \alpha = m + n$ (Knothe in 1957).
- Affine quermassintegral of order i. $\rightarrow \alpha = n - i$. (Lutwak in 1984).
- p-Minkowski addition with order i. $\rightarrow \alpha = p/(n - i)$. (Firey in 1962).
- Nilpotent Brunn-Minkowski in simply connected nilpotent Lie group of dimension n. $\rightarrow \alpha = n$. (Leonardi and Mansou in 2005).
- Newtonian and Logarithmic capacity. $\rightarrow \alpha = n - 2$ and $\alpha = 1$ (Borell in 1983, Caffarelli, Jerison and Lieb in 1996, Colesanti and Cuoghi in 2005).
- ... (See Gardner's survey article).

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- ... (See Gardner's survey article).
- P-capacity. $\rightarrow \alpha = n - p$, $1 < p < n$. (Colesanti and Salani in 2003).

Brunn-Minkowski inequality for p-Capacity

- p-capacity, $1 < p < n$, of a set convex body K is defined as

$$\text{Cap}_p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^p dx,\ v \in C_c^\infty(\mathbb{R}^n) :\ v \geq 1 \text{ on } K \right\}.$$

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If u is the minimizer then

$$\begin{cases} \Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\,\nabla u) = 0 \text{ in } \mathbb{R}^n \setminus K, \\ u = 1 \text{ on } \partial K, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

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- $\Delta_p u$ is a nonlinear generalization of the Laplace equation $\Delta u = 0$.
- $\text{Cap}(\cdot)$ is homogeneous of degree $n - p$.
- $\text{Cap}(\cdot)$ satisfies the Brunn-Minkowski inequality for convex bodies;

$$\left[\text{Cap}_p((1-\lambda)A + \lambda B)\right]^{\frac{1}{n-p}} \geq (1-\lambda)\left[\text{Cap}(A)_p\right]^{\frac{1}{n-p}} + \lambda\left[\text{Cap}(B)_p\right]^{\frac{1}{n-p}}.$$

Equality holds iff A is a homothetic copy of B.

$\mathcal{A}-$harmonic PDEs

More general nonlinear elliptic PDEs;

Let p be fixed, $1 < p < n$. Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$.

Suppose also that $\mathcal{A} = \mathcal{A}(\eta)$ has continuous partial derivatives in $\eta_k, 1 \leq k \leq n$, on $\mathbb{R}^n \setminus \{0\}$. We say that the function $\mathcal{A}$ belongs to the class $M_p(\alpha)$ if the following conditions are satisfied whenever $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$:

$$\begin{aligned} &(i)\ \alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial \mathcal{A}_i}{\partial \eta_j}(\eta)\xi_i\xi_j \leq \alpha|\eta|^{p-2}|\xi|^2, \\ &(ii)\ \mathcal{A}(\eta) = |\eta|^{p-1}\mathcal{A}(\eta/|\eta|). \end{aligned} \tag{1}$$

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Here $\{$Laplace's Eqn$\} \subsetneq \{$p-Laplace Eqns$\} \subsetneq \{\mathcal{A}$-harmonic PDEs$\}$.

$\mathcal{A}(\eta) = \eta$	$\mathcal{A}(\eta) = |\eta|^{p-2}\eta$	$\mathcal{A}(\eta)$ in (1).
Borell	Colesanti & Salani	??

Caffarelli & Jerison & Lieb

Brunn-Minkowski inequality for nonlinear Capacities

Given convex compact set K with $\mathcal{H}^{n-p}(K) = \infty$. Then there is u with

$$\begin{cases} \nabla \cdot \mathcal{A}(\nabla u) = 0 & \text{in} \quad \mathbb{R}^n \setminus K, \\ u = 1 & \text{on} \quad \partial K, \\ \lim_{|x|\to\infty} u(x) = 0. & \end{cases}$$

Define $\mathrm{Cap}_{\mathcal{A}}(K) := \int \langle \mathcal{A}(\nabla u), \nabla u \rangle dy.$

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Let $E_1, E_2 \subset \mathbb{R}^n$ be compact convex sets with $\mathcal{H}^{n-p}(E_i) = \infty$ then

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If equality holds and

(i) *There exists* $1\leq\Lambda<\infty$ *such that* $|\frac{\partial\mathcal{A}_i}{\partial\eta_j}(\eta)-\frac{\partial\mathcal{A}_i}{\partial\eta'_j}(\eta')|\leq\Lambda\,|\eta-\eta'||\eta|^{p-3}$

whenever $0<|\eta|\leq 2|\eta'|$ *and* $1\leq i\leq n$,

(ii) $\mathcal{A}_i(\eta)=\frac{\partial f}{\partial\eta_i}, 1\leq i\leq n$, *where* $f(t\eta)=t^p f(\eta)$, *when* $t>0, \eta\in\mathbb{R}^n\setminus\{0\}$,

then E_2 is a homothetic copy of E_1.

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- Step 1: If u is capacitary function for convex body E then

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Let E_1, E_2 be convex bodies and let u_1, u_2 be capacitary functions. Let u be the capacitary function for $E_1 + E_2$. Fix $\lambda \in (0,1)$, define

$$u^*(x) = \sup\left\{ \min\{u_1(y), u_2(z)\}\,;\ \begin{array}{l} x = \lambda y + (1-\lambda) z, \\ \lambda \in [0,1], y, z \in \mathbb{R}^n\,. \end{array} \right\}$$

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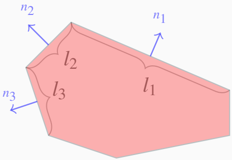
- Use Step 1; $\text{Cap}_{\mathcal{A}}(E_1+E_2)^{\frac{1}{p-1}} \ge \min(\text{Cap}_{\mathcal{A}}(E_1)^{\frac{1}{p-1}}, \text{Cap}_{\mathcal{A}}(E_2)^{\frac{1}{p-1}})$

which is the minimal capacitary Brunn-Minkowski inequality.

Minkowski Problem for Polyhedron

Fact: A convex polygon in $\mathbb{R}^2$ is uniquely determined (up to translation) by the unit normals $n_1, \ldots, n_m$ of the faces and lengths $l_1, \ldots, l_m$ of its edges.



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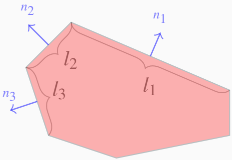

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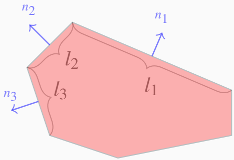

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Problem (Minkowski Problem for Polyhedron)
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Does there exists a convex Polyhedron $\mathcal{P} \subset \mathbb{R}^n$ whose faces have the given unit normals $\mathbf{n}_1, \ldots, \mathbf{n}_m$ and surface areas $A_1, \ldots, A_m$?

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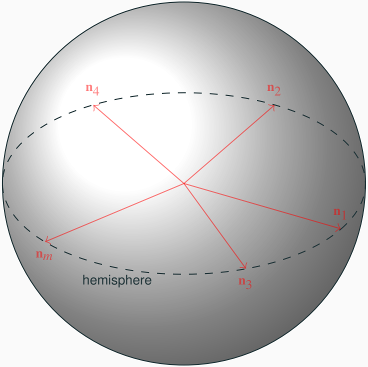

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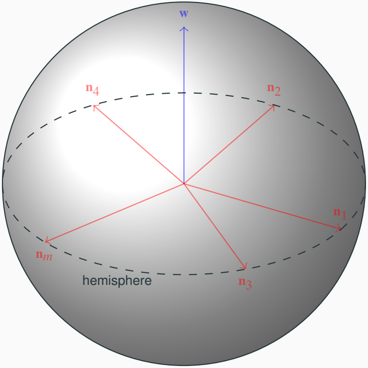

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Otherwise, there exists a unit vector $\mathbf{w}$ s.t. $\mathbf{w} \cdot \mathbf{n}_i = 0$ for every $i = 1, \ldots, m$.

Then Polyhedron will not be closed in the $-\mathbf{w}$ direction.

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- Both sides equal condition:

$$A_1 \mathbf{n}_1 + \ldots + A_m \mathbf{n}_m = 0.$$

There are n equations here so this imposes n conditions on A_i and $\mathbf{n}_i$.

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If $\mathbf{w} \in \mathbb{S}^{n-1}$ then the area of the projection of ith face to $\mathbf{w}^\perp$ is $A_i(\mathbf{n}_i \cdot \mathbf{w})$.

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Therefore, for every $\mathbf{w} \in \mathbb{S}^{n-1}$,

$$\sum_{i=1}^{m} A_i(\mathbf{n}_i \cdot \mathbf{w}) = \sum_{\mathbf{n}_i \cdot w > 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) + \sum_{\mathbf{n}_i \cdot w < 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) = 0.$$

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Therefore, for every $\mathbf{w} \in \mathbb{S}^{n-1}$,

$$\sum_{i=1}^{m} A_i(\mathbf{n}_i \cdot \mathbf{w}) = \sum_{\mathbf{n}_i \cdot w > 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) + \sum_{\mathbf{n}_i \cdot w < 0} A_i(\mathbf{n}_i \cdot \mathbf{w}) = 0.$$

- This condition fails if the Hemisphere condition does not hold.

Minkowski Problem for Polyhedron

Theorem (Solution to Minkowski Problem for Polyhedron)
Suppose that $\mathbf{n}_1, \ldots, \mathbf{n}_m \in \mathbb{S}^{n-1}$ *spans* $\mathbb{R}^n$ *and positive numbers* $A_1, \ldots, A_m$ *are given.*

Then there exists a Polyhedron $\mathcal{P}$ *whose faces have unit normals* $\mathbf{n}_1, \ldots, \mathbf{n}_m$ *and surface areas* $A_1, \ldots, A_m$ *if and only if*

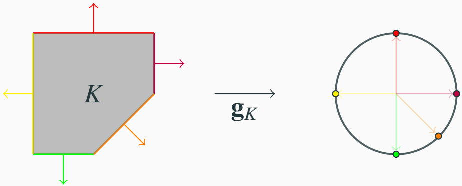
$$A_1\mathbf{n}_1 + \ldots + A_m\mathbf{n}_m = 0.$$

Moreover, this Polyhedron is unique up to translation.

- Due to Minkowski (1903).

Surface area measure on $\mathbb{S}^{n-1}$

- Let $\mathbf{g}_K = \mathbf{g} : \partial K \to \mathbb{S}^{n-1}$ be the Gauss map; $x \mapsto \mathbf{n}_x$. K is convex.



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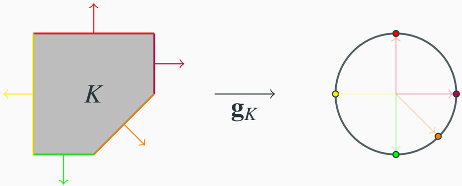

Consider the measure $d\mu_K = \mathbf{g}_*(d\mathcal{H}^{n-1})$ defined on the unit sphere by

$$\mu_K(E) = \int_{\mathbf{g}^{-1}(E)} d\mathcal{H}^{n-1} \quad \text{whenever} \quad E \subset \mathbb{S}^{n-1} \text{ is Borel.}$$

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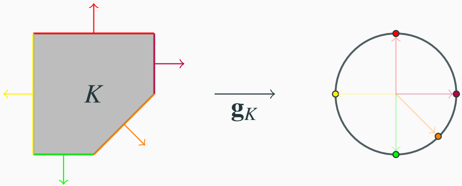

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- When $K = \mathcal{P}$ is convex Polyhedron with unit normals

$\mathbf{n}_1, \ldots, \mathbf{n}_m \in \mathbb{S}^{n-1}$ and surface areas $A_1, \ldots, A_m$ then

$$d\mu_{\mathcal{P}} = \sum_{i=1}^{m} A_i \delta_{\mathbf{n}_i} \quad \text{where } \delta_{\mathbf{n}_i} \text{ is Dirac point mass measure at } \mathbf{n}_i.$$

Surface area measure on $\mathbb{S}^{n-1}$

- If $\partial K \in C^2$ and has positive Gauss curvature everywhere then

$$d\sigma_K(X) = \frac{1}{\kappa(X)} d\mathcal{H}^{n-1}(X) \quad \text{on} \quad \mathbb{S}^{n-1}$$

where $\kappa(X)$ is the Gauss curvature at the point of ∂K where X is the outer unit normal to ∂K.

- Gauss curvature κ is the Jacobian determinant of the Gauss map $\mathbf{g}$.

Minkowski Problem – General Case

Problem (Minkowski Problem)
Given a positive Borel measure μ on $\mathbb{S}^{n-1}$, does there exists a convex body K in $\mathbb{R}^n$ with surface area measure μ_K such that $\mu_K = \mu$?

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Necessary conditions:

- **C1**: the centroid of the measure μ is at the origin;

$$\int_{\mathbb{S}^{n-1}} w \, d\mu(w) = 0.$$

This is due to μ is translation invariant.

- **C2**: μ is not supported on any equator of $\mathbb{S}^{n-1}$;

$$\int_{\mathbb{S}^{n-1}} |\theta \cdot w| d\mu(w) > 0 \quad \text{for every} \quad \theta \in \mathbb{S}^{n-1}.$$

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- K is unique up to translation.
- These results are due to Minkowski, Alexandrov, Fenchel-Jessen.

Minkowski Problem - General case

Theorem
Let μ be a non-negative Borel measure on $\mathbb{S}^{n-1}$ satisfying

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- **Existence**: *There is a convex body K with non-empty interior such that $\mu_K = \mu$.*
- **Uniqueness**: *K is unique up to translation.*
- **Regularity**: *If $d\mu_K = \frac{1}{\kappa} d\mathcal{H}^{n-1}$ for some strictly positive function $\kappa \in C^{k,\alpha}(\mathbb{S}^{n-1})$ for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$ then K is $C^{k+2,\alpha}$.*

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► Existence is solved by Minkowski (1903) for the case of polyhedron, in general Alexandrov (1937-1938), Fenchel and Jessen (1938).
► C^∞ regularity is proved by Lewy (1938), Pogorelov (1953), Nirenberg (1953), Cheng and Yau (1976).
► The precise gain of two derivatives and the treatment of small values of k due to Caffarelli (1990).

The support function of a convex body

- The support function h_K of a convex domain K in $\mathbb{R}^n$ is defined as

$$h_K : \mathbb{S}^{n-1} \to \mathbb{R}, \quad h_K(X) = \sup\{\langle X, x\rangle; x \in K\}$$

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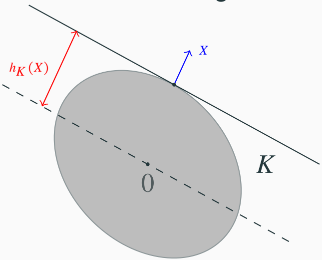

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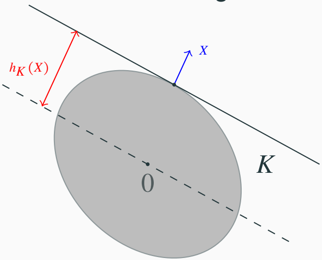

-h_K is homogeneous of degree 1.
-Any non-empty closed convex set K is uniquely determined by h_K.

For every convex bodies K, L and constants $\alpha, \beta \geq 0$;

$$h_{\alpha K+\beta L} = \alpha h_K + \beta h_L$$

where

$$A + B := \{a + b \mid a \in A,\, b \in B\}\,.$$

The Hadamard Variational Formula

If K is convex body with support function h_K and L is any other convex body with support function h_L. Then

$$\mathcal{M}(K,L) := \lim_{\epsilon\to 0^+} \frac{|K+\epsilon L| - |K|}{\epsilon} = \int_{\mathbb{S}^{n-1}} h_L(X)\, d\mu_K(X) = \int_{\partial K} h_L(\mathbf{g}(x))\, d\mathcal{H}^{n-1}.$$

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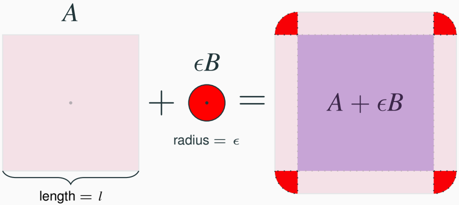

$$|A+\epsilon B| = |A| + 4l\epsilon + |\epsilon B|.$$

$$\lim_{\epsilon\to 0} \frac{|A+\epsilon B| - |A|}{\epsilon} = \frac{4l\epsilon + \epsilon^2|B|}{\epsilon}$$

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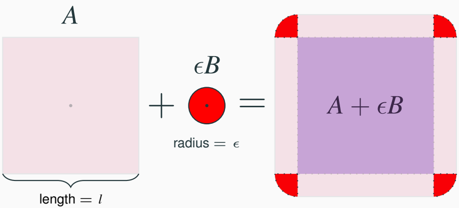

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- The first variation $\mathcal{M}(K,L)$ of the Lebesgue measure is the surface area measure μ_K.

A representation formula for volume

$$\mathcal{M}(K,L) = \lim_{\epsilon \to 0^+} \frac{|K+\epsilon L| - |K|}{\epsilon} = \int_{\mathbb{S}^{n-1}} h_L(X)\, d\mu_K(X).$$

If we take $L = K$ in $\mathcal{M}(K,L)$ then

$$\int_{\mathbb{S}^{n-1}} h_K(X)\, d\mu_K(X) = \lim_{\epsilon \to 0^+} \frac{|K+\epsilon K| - |K|}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{|(1+\epsilon)K| - |K|}{\epsilon}$$

$$= |K| \lim_{\epsilon \to 0^+} \frac{(1+\epsilon)^n - 1}{\epsilon} = n|K|.$$

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Hence $\quad |K|=\dfrac{1}{n}\displaystyle\int_{\mathbb{S}^{n-1}} h_K(X)\,d\mu_K(X)=\int_{\partial K} h_K(\mathbf{g}(x))\,d\mathcal{H}^{n-1}=\frac{1}{n}\mathcal{M}(K,K)$

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- Using $(n-1)-$homogeneity of surface area measure, a re-scaled copy K of $\tilde{K}$ provides a solution.

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The variational formula for volume and $\mu_K = \mu_L$ gives

$$\begin{aligned} F'(0) = \lim_{t \to 0^+} \frac{F(t) - F(0)}{t} &= \frac{1}{n}|L|^{-\frac{n-1}{n}} \frac{d}{dt}\left(|tK + (1-t)L|\right)\big|_{t=0} \\ &= \frac{1}{n}|L|^{-\frac{n-1}{n}} \int_{\mathbb{S}^{n-1}} \left(h_K(X) - h_L(X)\right) d\mu_L(X) \\ &= \frac{1}{n}|L|^{-\frac{n-1}{n}} \left[|K| - |L|\right] = [F(0)]^{1-n}\left[F(1)^n - F(0)^n\right]. \end{aligned}$$

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Hence $F(t) =$ constant for $t \in [0,1]$;

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- Therefore K and L are homothetic.
- $\mu_K = \mu_L$ implies K and L have same perimeter $\Rightarrow$ K is at most translation of L.

Minkowski type problems for other measures

- L_p Minkowski problem (L_0 is the usual Minkowski problem) due to Andrews in 1999, Chou and Wang in 2006, Hug, Lutwak, Yang, and Zhang in 2005, Ludwig in 2011, Lutwak and V. Oliker in 1995.
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- Electrostatic Capacitary surface measure associated to Laplace's equation; existence due to Jerison in 1996, uniqueness due to Borell in 1984 and Cafarelli, Jerison, Lieb in 1996.

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Indeed, u has the asymptotic expansion

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Moreover, $G(x) = |x|^{2-n}$ is the fundamental solution to $\Delta u = 0$;

$$\Delta G(x) = -\delta_0.$$

Minkowski problem for electrostatic capacity

Classical Minkowski problem; $d\mu_K = \mathbf{g}_*(d\mathcal{H}^{n-1})$ defined on the unit sphere by

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Problem

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Theorem (Jerison in 1996)
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Also the resemblance between the variation of the capacity

$$\frac{d}{dt}\mathrm{Cap}_2(K + tK_1)|_{t=0} = \int_{\partial K} h_{K_1} |\nabla u_K|^2 \, d\mathcal{H}^{n-1}$$

and the variation of the volume $\quad \frac{d}{dt}|K + tK_1|\,|_{t=0} = \int_{\partial K} h_{K_1} \, d\mathcal{H}^{n-1}.$

Sketch of Uniqueness

As in the classical Minkowski problem, existence relies on

- The Hadamard variational formula for capacity.
- The Brunn-Minkowski inequality for capacity.

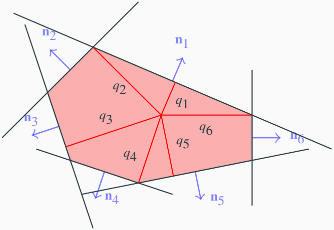
Use these two to show that $m(t)$ is constant for $t \in [0, 1]$ where

$$m(t) = \left(\mathrm{Cap}((1-t)E_1 + tE_2)\right)^{\frac{1}{n-2}}.$$

Sketch of Existence

- Let $\mathbf{n}_1, \ldots, \mathbf{n}_m$ be unit normals which spans $\mathbb{R}^n$ and let $c_1, \ldots, c_m$ be positive numbers such that $\sum_{i=1}^{m} c_i \mathbf{n}_i = 0 \to \mu = \sum_{i=1}^{m} c_i \delta_{\mathbf{n}_i}$.

Sketch of Existence



Given $q = (q_1, \ldots, q_m) \in \mathbb{R}^m$ with $q_i \geq 0$.

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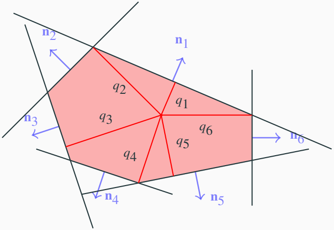

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Consider the following minimization problem

$$\inf \left\{ \sum_{i=1}^{m} c_i q_i;\ \mathcal{P}(q) \in \Theta \right\} \quad \text{where} \quad \Theta = \{\mathcal{P}(q);\ \text{Cap}(\mathcal{P}(q)) \geq 1\}.$$

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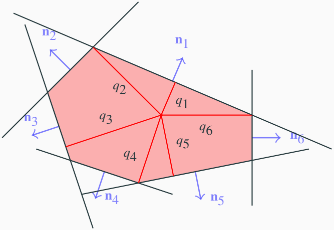

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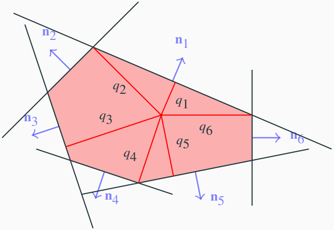

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Then $\text{Cap}(\mathcal{P}(\tilde{q})) = 1$ and $\mathcal{P}(\tilde{q})$ has faces F_k with outer unit normal $\mathbf{n}_k$;

$$\mu(\mathbf{n}_k) = c_k = \lambda \mu^2_{\mathcal{P}(\tilde{q})}(\mathbf{n}_k) \quad \text{for } k = 1, \dots, m, \quad c_k = \frac{\lambda}{n-2} \int_{F_k} |\nabla u|^2 \mathcal{H}^{n-1}.$$

The Minkowski problem for p-capacitary surface measures

Let u be the p-capacitary function $(1 < p < n)$ for convex body K

$$\begin{cases} \Delta_p u = 0 & \text{in} \quad \mathbb{R}^n \setminus K, \\ u = 1 & \text{on} \quad \partial K, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

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- *K is unique up to translation when $p \neq n-1$ (and is unique up to translation and dilation when $p = n-1$).*
- *This problem has been considered by Colesanti, Nyström, Salani, Xiao, Yang, Zhang under additional assumptions on μ with for $1 < p < 2$.*
- *We prove the same result for capacitary surface measures associated to $\mathcal{A}-$harmonic PDEs.*

THANKS!

Regularity of Minkowski Problem - Monge-Ampère equation

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- Equivalent statement of Minkowski problem: Find a closed, convex hypersurface K whose Gaussian curvature prescribed as a positive function defined on $\mathbb{S}^{n-1}$.