Absolute continuity of harmonic measure, interior approximation, and cone points

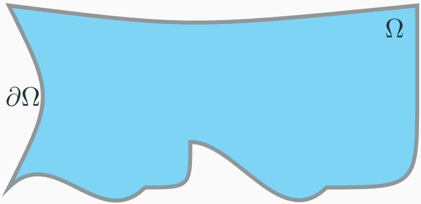
Murat Akman

December 13

Analysis and PDE Seminar - Worcester Polytechnic Institute

Dirichlet Problem and Harmonic measure

- Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, connected and open.



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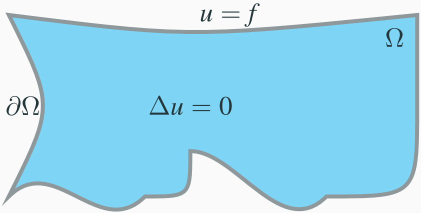

Dirichlet problem:

$$(\text{D}) \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \\ u \in C^2(\Omega) \cap C(\partial\Omega) \\ f \in C_c(\partial\Omega). \end{cases}$$

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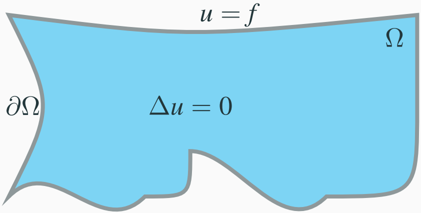

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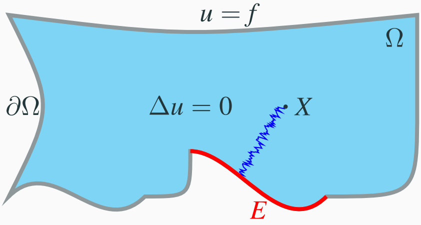

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- Harmonic measure $\omega_\Omega^X(E)$ of E with a given pole X is the probability that a Brownian motion starting at X will first hit $\partial\Omega$ in the set E.

Examples of Harmonic Measure

- If $\Omega = \mathbb{B}^{n+1}$, $(n+1)-$dimensional unit ball, and $X \in \Omega$. Then

$$\omega^X(E) = \frac{1}{\mathcal{H}^n(\mathbb{S}^n)} \int_E \frac{1 - |X|^2}{|X - Y|^{n+1}} \, d\mathcal{H}^n(Y) \quad \text{for every Borel set } E \subset \mathbb{S}^n.$$

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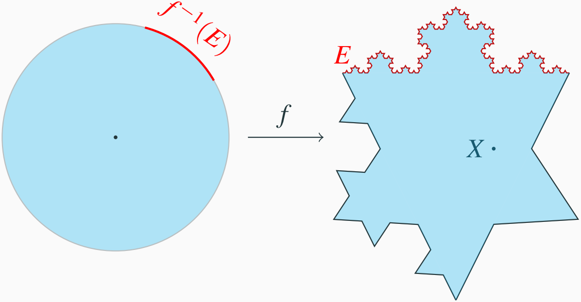
- If $\Omega \subset \mathbb{R}^{n+1}$ is bounded domain of class C^1, then there is $K(X,Y) : \Omega \times \partial\Omega \to \mathbb{R}$ such that

$$\omega^X(E) = \int_E K(X,Y)\, d\mathcal{H}^n(Y) \quad \text{for every Borel set } E \subset \partial\Omega.$$

More examples of Harmonic Measure

- If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, $\partial\Omega$ is Jordan curve then



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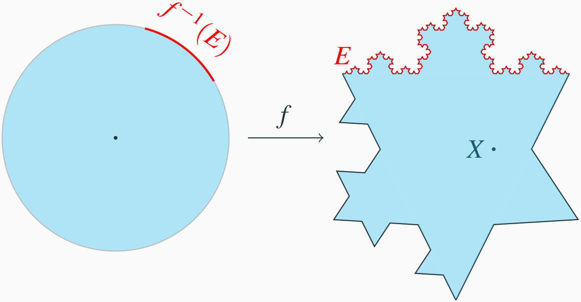

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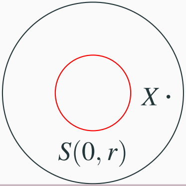
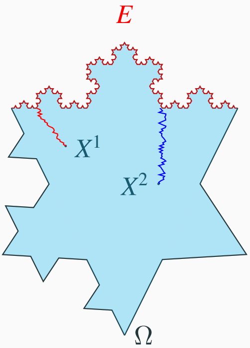
- If $\Omega = A(0, r, R) \subset \mathbb{R}^{n+1}$ is an annular region then the harmonic measure of the inner shell $S(0, r)$ is

$$\omega^X(S(0,r)) = \begin{cases} \frac{\log R - \log |X|}{\log R - \log r} & \text{if } n = 1, \\ \frac{|X|^{2-(n+1)} - R^{2-(n+1)}}{r^{2-(n+1)} - R^{2-(n+1)}} & \text{if } n \geq 2. \end{cases}$$



Harmonic Measure at Different Poles



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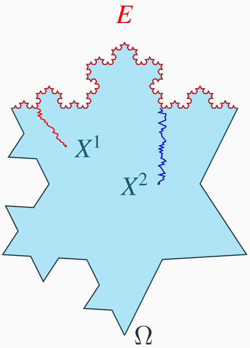

- For every borel set $E \subset \partial\Omega$, $X \to \omega^X(E)$ is a non-negative harmonic function in Ω.
- Harmonic measure ω^{X_1} and ω^{X_2} at different poles are mutually absolutely continus; $\omega^{X_1}(E) = 0 \Leftrightarrow \omega^{X_2}(E) = 0$.

$c^{-1}\omega^{X_1}(E) \leq \omega^{X_2}(E) \leq c\omega^{X_1}(E)$.

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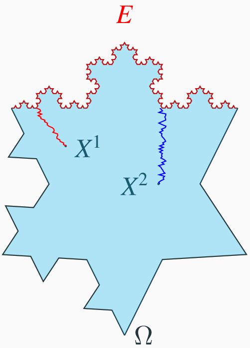

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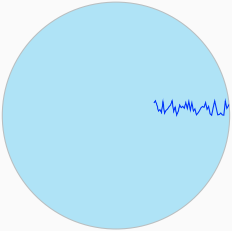
- **Drop the pole** X to get the **harmonic meeasure** ω of Ω.

Detecting the exit point of a Brownian motion

Consider a random Brownian particle moving in a domain $\Omega \subset \mathbb{R}^2$.

► Aim is to find the point where it first hits the boundary $\partial\Omega$.



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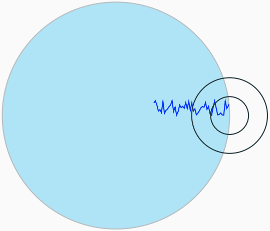

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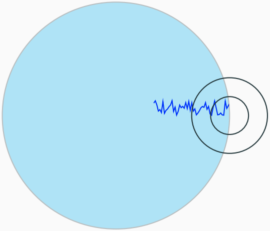

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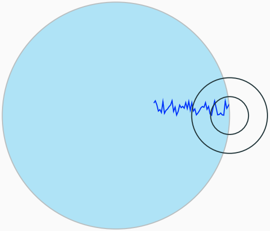

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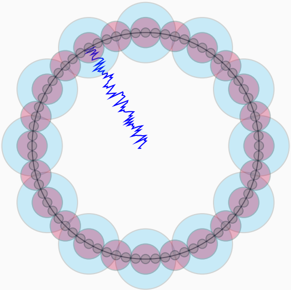
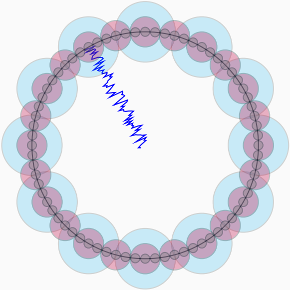


▶ Note that to detect an exit at x, the point must be contained in infinitely many detectors whose radii tend to zero.

Detecting the exit point of Brownian motion

- When Ω is the unit disk $\mathbb{D}$, and the Brownian particle starts at 0 then the hitting distribution on $\partial\Omega$ is normalized Lebesgue measure.
- Thus to detect the exit point almost surely, we must cover almost every point of $\partial\Omega$ by arbitrarily small balls.



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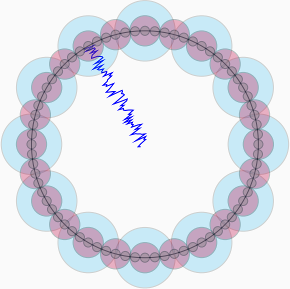

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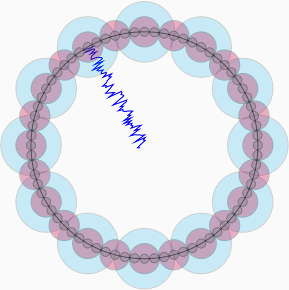
- If $\phi(r) \geq r$ then we can not detect the exit point on a finite budget.
- However, if $\phi(r) = o(r)$ then we can cover $\partial\Omega$ by about n_k balls of size $1/n_k$ and let $n_k \nearrow \infty$ so fast that $\sum n_k \phi(1/n_k) < \infty$.



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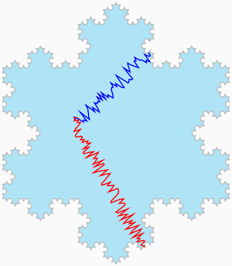

- If $\partial\Omega$ is the von Koch Snowflake then it takes roughly 4^n balls of size 3^n to cover the whole boundary, which we can do on a finite budget iff $\phi(t) = o(t^\alpha)$, where $\alpha = \log 4/\log 3 > 1$.

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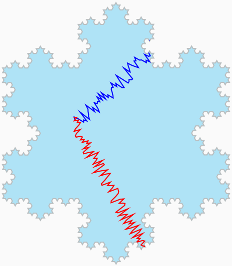

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- However, not all parts of the snowflake are equally likely to be hit by Brownian motion, and there is a small subset of $\partial\Omega$ which still gets hit with probability 1.

Hausdorff measure and Hausdorff dimension

Given increasing function ϕ on $[0, \infty)$, define the Hausdorff ϕ-measure of E

$$\mathcal{H}_\phi(E) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \phi(r_i);\ E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \leq \delta \right\}.$$

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- $\mathcal{H}^2$ is multiple of Lebesgue area measure; $\mathcal{H}^1$ is length...

$\mathcal{H}^n_\infty(E)$ is called the Hausdorff content of E and is defined as

$$\mathcal{H}^n_\infty(E) = \inf \left\{ \sum_{i=1}^{\infty} (r_i)^n;\; E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}.$$

- $\mathcal{H}^\alpha_\infty(E) \leq \mathcal{H}^\alpha_\delta(E) \leq \mathcal{H}^\alpha(E)$. But still $\mathcal{H}^\alpha_\infty(E) = 0 \iff \mathcal{H}^\alpha(E) = 0$.

Being singular $\perp$ and absolutely continuous $\ll$

The Hausdorff dimension of a set E is defined by

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The smaller α is, the more expensive it is to cover E; the dimension marks the transition from positive to zero cost coverings.

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The dimension of a measure μ is the smallest dimension of a full μ-measure set, i.e.,

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E):\ \mu(E^c) = 0\} = \inf\{\alpha:\ \mu \perp \mathcal{H}^\alpha\}$$

- $\mu \perp \nu$ if there is a set E such that $\mu(E) = \nu(E^c) = 0$
- $\mu \ll \nu$ if $\nu(E) = 0 \Rightarrow \mu(E) = 0$.
- $\mu \sim \nu$ if $\nu \ll \mu \ll \nu$.

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It is always true that

$$\dim_{\mathcal{H}}(\mu) \leq \dim_{\mathcal{H}}(\mathrm{supp}(\mu)).$$

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We can detect a.e. exit point of Brownian motion on a finite ϕ-budget iff there is a Vitali covering of a full ω-measure set E by disks of radius $\{r_j\}$ such that $\sum \phi(r_j) < \infty$.

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Thus the detection question is really:

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- $n - 1 \leq \dim_{\mathcal{H}}(\omega) \leq n + 1 - c_n$ for some dimensional constant $c_n > 0$.

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- **Lavrentiev**('36): Quantitative version.

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- **Wu**('86): $\omega \not\ll \mathcal{H}^2$ for some topological sphere in $\mathbb{R}^3$.
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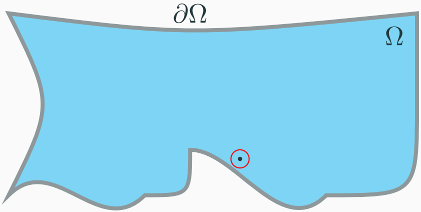
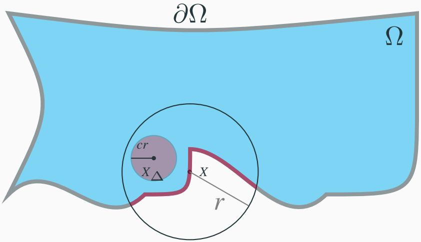
Non-tangentially Accessible Domains(NTA)

- Openness



Non-tangentially Accessible Domains(NTA)

- Openness $\rightsquigarrow$ Corkscrew condition.



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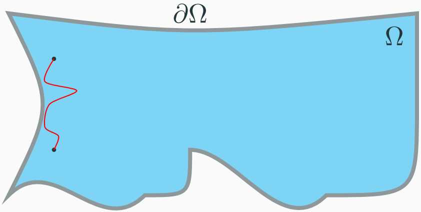

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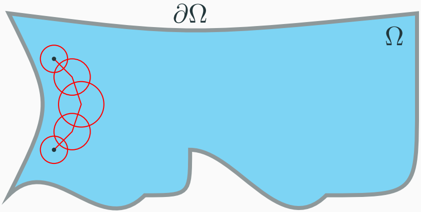

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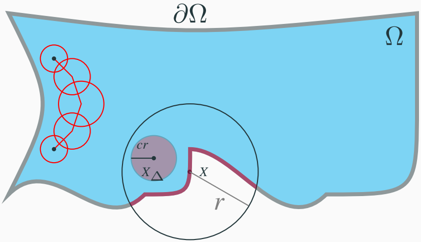

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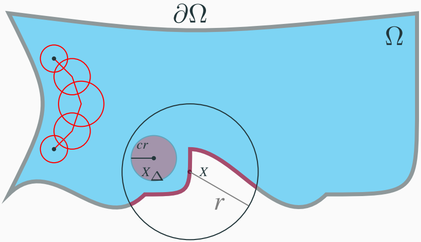

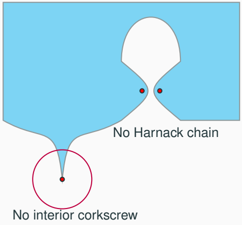

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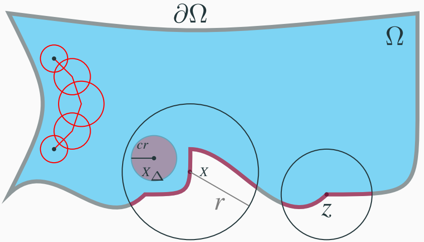

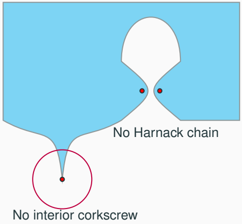

- Ω is NTA $\equiv$
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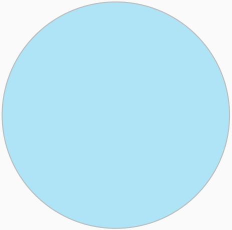
- $\partial\Omega$ is $n-$Ahlfors-David regular (ADR) if

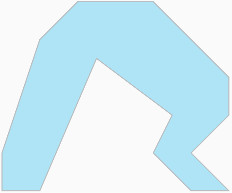
$cr^n \leq \mathcal{H}^n(\partial\Omega \cap B(z,r)) \leq cr^n$ whenever $z \in \partial\Omega$ and $r \in (0, \text{diam}(\partial\Omega))$.

Examples of such domains

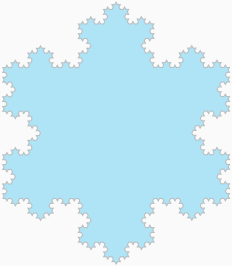
Smooth Domains



Lipschitz Domains



NTA Domains



- NTA domains need not be graph domains or of finite perimeter.

Global results in higher dimension

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- **Azzam, Mourgoglou, and Tolsa**('15): $\exists$ NTA domain Ω with $\mathcal{H}^n(\partial\Omega) < \infty$ such that $\omega \not\ll \mathcal{H}^n|_{\partial\Omega}$ (Using the deep result of Wolff which was further developed by Lewis, Nyström, Vogel).

Necessary conditions for Absolute Continuity

- **Pommerenke**('86): If $\Omega \subset \mathbb{C}$ is simply connected and $\omega \ll \mathcal{H}^1$ on a set $E \subset \partial\Omega$ then ω a.e. point E is a cone point for Ω and $\omega-$almost all of F can be covered by a countable union of $n-$dimensional (possibly rotated) Lipschitz graphs (i.e., $\omega|_E$ is rectifiable).

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Theorem (Azzam-Mourgoglou-Tolsa $\bigoplus$ Hofmann-Martell-Mayboroda-Tolsa-Volberg, ('15))

- *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$,* **open** *and* **connected**.
- *Let $F \subset \partial\Omega$ with $\mathcal{H}^n(F) < \infty$.*

1. *If $\omega \ll \mathcal{H}^n$ on $F \implies \omega|_F$ is $n-$rectifiable.*
2. *If $\mathcal{H}^n \ll \omega$ on $F \implies F$ is $n-$rectifiable.*

⋆ Portion of the boundary should be contained in a nice rectifiable set(like a graph or curve)!

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A set $\Omega \subset \mathbb{R}^{n+1}$ has **big boundary** if

$$\mathcal{H}^n_\infty(B(z,r) \setminus \Omega) \geq cr^n \quad \text{for all } z \in \partial\Omega \text{ and } r \in (0, \text{diam}(\partial\Omega)).$$

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Suppose $\Omega \subset \mathbb{R}^{n+1}$ has big boundary and let $\Gamma \subset \mathbb{R}^{n+1}$ is $n-$ADR and splits $\mathbb{R}^{n+1}$ into two NTA domains. Then $\omega_\Omega \ll \mathcal{H}^n$ on $\partial\Omega \cap \Gamma$.

Second result

Theorem (A., Azzam, Mourgoglou ('16))

Suppose $\Omega \subset \mathbb{R}^{n+1}$ *has big boundary and* Γ *is a Lipschitz graph.*
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Moreover, if F *is the set of cone points in* $\partial\Omega$*, then*

$$\omega \sim \mathcal{H}^n \quad \text{on} \quad F.$$

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A closed set $E \subset \mathbb{R}^{n+1}$ is called **uniformly p-fat** if

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Let Ω have big boundary and Γ be ADR splits $\mathbb{R}^{n+1}$ into two NTA domains Ω_1, Ω_2. Aim: $E \subset \Gamma \cap \partial\Omega$, show $\omega_{\Omega}^{X_0}(E) > 0 \Rightarrow \mathcal{H}^n(E) > 0$.

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Then, by strong Markov property of Brownian motion, for $X \in \Omega \cap \Omega_1$

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Same holds for $X \in \Omega \cap \Omega_2$ and hence

$$\sup_{X \in \Omega} \omega_{\Omega}^{X}(E) \leq \gamma < 1 \text{ which is NOT possible!}$$

Sketch of the Proof cont'

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$$B^i = B(Y_i, cr) \subset \Omega_i \cap B(X, r) \text{ for } i = 1, 2.$$

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$$\omega_{\Omega \cap \Omega_i}^{Y_i}(\Gamma \cap \Omega) < \eta \text{ for some } \eta \in (0, 1) \text{ and } i \in \{1, 2\}.$$

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If so, the Harnack chain, and ω_Ω is probability measure imply that

$$\begin{aligned}
\omega_\Omega^X(E) &\leq (1-t) + t\omega_\Omega^{Y_i}(E^c) \\
&= (1-t) + t\left(\omega_{\Omega \cap \Omega_i}^{Y_i}(E) + \int_{\partial\Omega_i \cap \Omega} \omega_\Omega^Z(E)\, d\omega_{\Omega \cap \Omega_1}^{Y_i}(Z)\right) \\
&< (1-t) + t(0 + \eta) = (1-t) + t\eta =: \gamma < 1.
\end{aligned}$$

Sketch of the Proof cont'

Hence, we need to show

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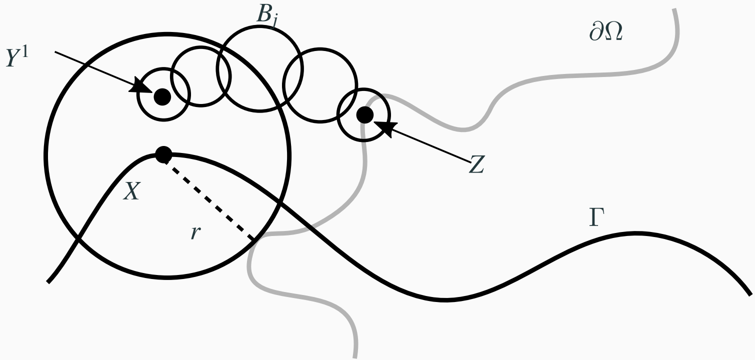
So we focus on proving $\omega_{\Omega\cap\Omega_i}^{Y_i}(\Gamma \cap \Omega) < \eta$.

Sketch of the Proof cont'

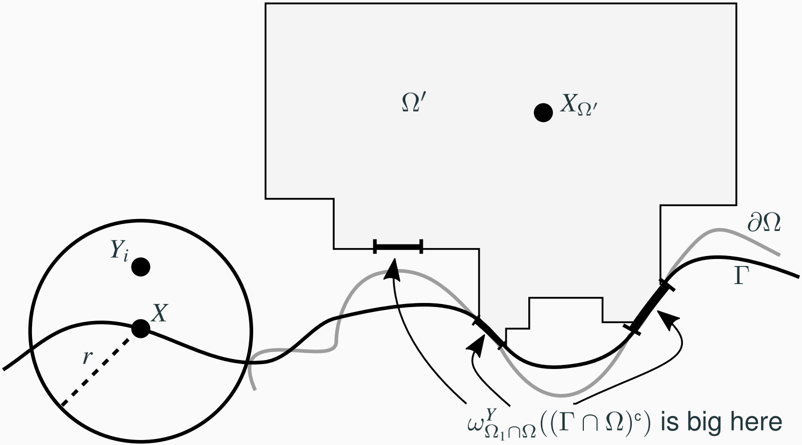
Proof of $\omega_{\Omega\cap\Omega_i}^{Y_i}(\Gamma\cap\Omega) < \eta$. Let $M_0 >> 1$.

Case 1: There is $Z \in \partial\Omega \cap B(X, M_0 r) \cap \Omega_1$ so that $\mathrm{dist}(Z, \Gamma) \geq \epsilon r$



In this case, Brownian motion starting at Y^1 has a good chance of hitting outside $\Gamma \cap \partial\Omega$.

Case 2: dist$(Z, \Gamma) \le \epsilon r$ for all $Z \in \partial\Omega \cap B(X, M_0 r) \cap \Omega_1$.



If black parts are G then we can pick i so that $\mathcal{H}^n(G) \ge \mathcal{H}^n(\partial\Omega')$.
Then result of David and Jerison implies

$$1 \lesssim \omega^{X_{\Omega'}}_{\Omega'}(G) \lesssim \omega^{X_{\Omega'}}_{\Omega_1\cap\Omega}((\Gamma\cap\Omega)^{c}) \lesssim \omega^{Y}_{\Omega_1\cap\Omega}((\Gamma\cap\Omega)^{c})$$

This gives $\omega^{Y}_{\Omega_1\cap\Omega}(\Gamma\cap\Omega) < \eta$.

Final Touch

- Take Wu's work as a starting point.
- Use good properties of Sawtooth constructions of Hofmann and Martell.
- Use tangent measure techniques introduced by Preiss ('87), developed for harmonic measure by Kenig and Toro (2006), Kenig, Preiss, and Toro ('09).

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- Use tangent measure techniques introduced by Preiss ('87), developed for harmonic measure by Kenig and Toro (2006), Kenig, Preiss, and Toro ('09).
- First part of our work also holds locally, and applies for a class of Elliptic measures considered by Kenig and Pipher ('01) and A., Badger, Hofmann, Martell ('15).

Thanks!