
7.4 Integration by Partial Fractions

Remember:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \text{ if } a \neq b,$$

$$\frac{1}{x(x^2+a)} = \frac{A}{x} + \frac{Bx+C}{x^2+a} \text{ if } a \neq 0,$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \text{ if } a \neq 0.$$

Example. Evaluate $\int \frac{2x+1}{x^2-4} dx$.*Thinking about the problem:*

The integrand $\frac{2x+1}{x^2-4}$ is a rational function and does not look like it can be handled with substitution, so we use partial fractions. The denominator x^2-4 is $(x+2)(x-2)$, a product of different linear factors, so the partial fraction decomposition of $\frac{2x+1}{x^2-4}$ is $\frac{A}{x+2} + \frac{B}{x-2}$ for some constants A and B . After solving for A and B we would have $\int \frac{2x+1}{x^2-4} dx = \int \frac{A}{x+2} dx + \int \frac{B}{x-2} dx$ and can integrate the right side.

Doing the Problem:

Writing $\frac{2x+1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$, solve for A and B by multiplying both sides by the denominator x^2-4 :

$$2x+1 = A(x-2) + B(x+2)$$

. Setting $x = 2$, we find

$$2(2)+1 = 5 = A(0) + B(2+2) = 4B \Rightarrow 5 = 4B \Rightarrow B = \frac{5}{4}.$$

Setting $x = -2$, we find

$$2(-2) + 1 = A(-2 - 2) + B(0) \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$$

Therefore $\frac{2x + 1}{x^2 - 4} = \frac{3/4}{x + 2} + \frac{5/4}{x - 2}$, so

$$\begin{aligned} \int \frac{2x + 1}{x^2 - 4} dx &= \frac{3}{4} \int \frac{dx}{x + 2} + \frac{5}{4} \int \frac{dx}{x - 2} \\ &= \frac{3}{4} \ln |x + 2| + \frac{5}{4} \ln |x - 2| + C. \end{aligned}$$

Solutions should show all of your work, not just a single final answer.

1. Evaluate $\int \frac{dx}{x^2 - 4x + 3}$.

(a) Decompose the integrand into partial fractions.

(b) Evaluate the integral using partial fractions.

2. Evaluate $\int \frac{x^2 + x + 1}{x(x^2 + 9)} dx$.

(a) Decompose the integrand into partial fractions.

(b) Evaluate the integral using partial fractions.

3. Evaluate $\int \frac{x+1}{x^3-8x^2+20x} dx$.

4. T/F (with justification): Computing $\int \frac{x}{x^2-1} dx$ requires partial fractions.