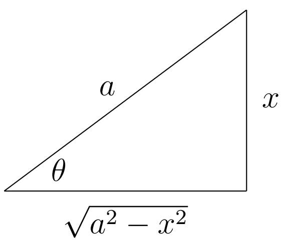
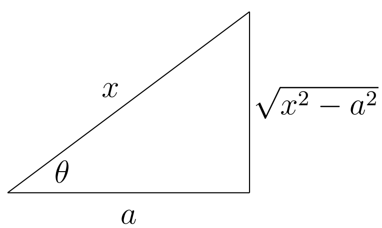
7.3 Trigonometric Substitution

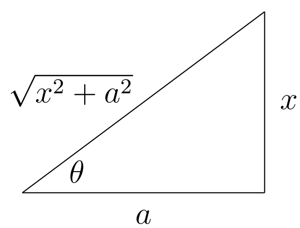
In each of the following trigonometric substitution problems, draw a triangle and label an angle and all three sides corresponding to the trigonometric substitution you select.

Summary of Trigonometric Substitution (understand how to make the triangles).

$\sqrt{a^2 - x^2}$ | $\sqrt{x^2 - a^2}$ | $\sqrt{x^2 + a^2}$







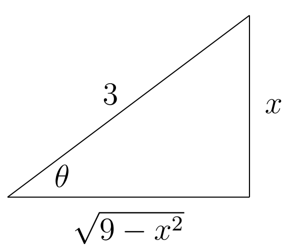
$\sin\theta = \frac{x}{a}$ $\qquad$ $\cos\theta = \frac{a}{x}$ $\qquad$ $\tan\theta = \frac{x}{a}$

Keep in mind the identities $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$.

Example: Evaluate $\int \frac{dx}{\sqrt{9 - x^2}}$.

Thinking about the problem:

Since the integrand involves $\sqrt{9 - x^2}$ and there is not an extra factor of x in the numerator (if there were it might be possible to do a u-substitution with $u = 9 - x^2$), we will try a trigonometric substitution corresponding to a right triangle with a leg of length $\sqrt{9 - x^2}$, hypotenuse 3, and the other leg has length x.



Doing the problem:

Using the triangle diagram above, $\sin\theta = x/3$, so $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$. Also from the triangle $\cos\theta = \sqrt{9-x^2}/3$, so $\sqrt{9-x^2} = 3\cos\theta$. The integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3\cos\theta\, d\theta}{3\cos\theta}\\ &= \int d\theta\\ &= \theta + C.\end{aligned}$$

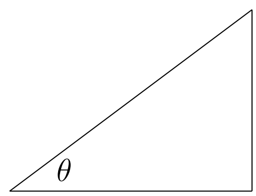
Since the substitution we used was $x = 3\sin\theta$, $\theta = \arcsin\left(\frac{x}{3}\right)$. So

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\left(\frac{x}{3}\right) + C.$$

Solutions should show all of your work, not just a single final answer.

1. Evaluate $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}$.

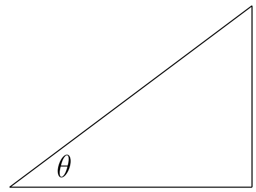
   (a) Fill in the sides of the right triangle below where $\sqrt{9+x^2}$ is one of the sides.



   (b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x.

2. Evaluate $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$.

(a) Fill in the sides of the right triangle below where $\sqrt{x^2-9}$ is one of the sides.



(b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x.

3. Evaluate the definite integral $\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$. (*Hint:* When you make a trigonometric substitution, change the bounds of integration as part of the substitution.)

4. T/F (with justification): To evaluate $\int \frac{dx}{x^2\sqrt{x^2+2}}$ by trigonometric substitution, use $x = 2\tan\theta$.