6.4 Work

If a constant force F applied to an object moves it along a straight line distance d then the **work** done on the object is $W = Fd$. If a varying force applied to an object moves it along a line from $x = a$ to $x = b$ then the **work** done on the object is the integral $W = \int_a^b f(x)\,dx$.

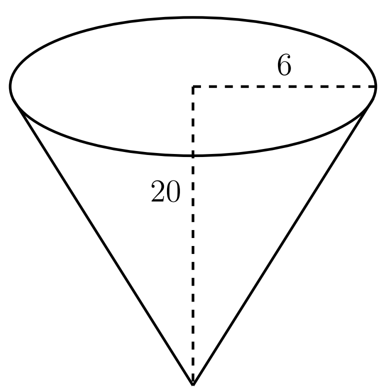
Two special cases are spring problems and lifting/pumping problems.

Force formula for springs. Hooke's law says the force required to maintain a spring stretched x units beyond its natural resting length is $F(x) = kx$, where k is a positive constant called the spring constant. It varies from spring to spring.

Work formula for tanks. The work required to pump out the liquid in a tank from its top, between levels a and b, is $\int_a^b \rho g D(y)A(y)\,dy$, where the slice at y has depth $D(y)$ from the top, cross-sectional area $A(y)$, and thickness dy. The density of the liquid is assumed to be a constant ρ (Greek "rho") and g is the acceleration due to gravity (another constant).

Example. An inverted conical tank with radius 6 feet and height 20 feet is full of water. Find the work required to pump the water out of the top of the tank. Use the fact that water weights 62.5 lb/ft^3.

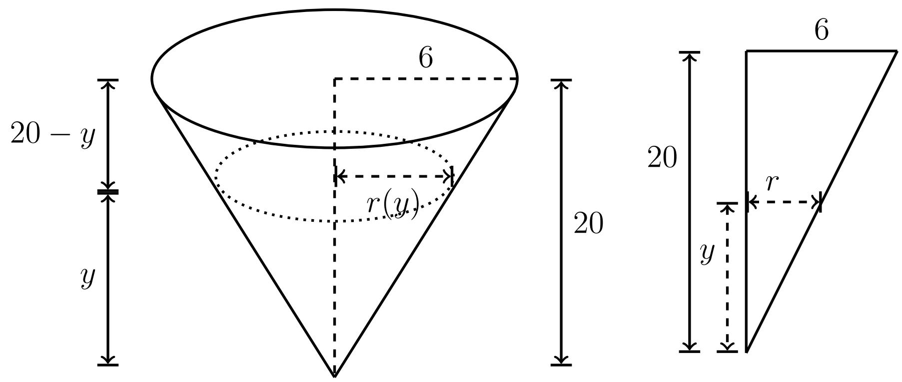
Thinking about the problem: We start by drawing a diagram.



From the problem's statement we will find a, b, ρ, g, $D(y)$, and $A(y)$ to get the definite integral for the work. It units are (force)(distance) = pound-feet, or foot-pounds.

Doing the problem:

Let y be the height of a slice from the bottom, so $y = 0$ is the bottom and a slice at height y has depth $D(y) = 20 - y$ from the top. Emptying the whole tank means we will integrate from the bottom ($y = 0$) to the top ($y = 20$), so $a = 0$ and $b = 20$.



To find the y-slice's area $A(y)$, it is a circle with radius $r(y)$, say, so its area is $\pi r(y)^2$. To figure out $r(y)$ in terms of y we use similar right triangles (figure above on the right). Comparing similar triangles shows $r/y = 6/20$, so $r = (6/20)y = (3/10)y$. Therefore the area of the y-slice is $A(y) = \pi r(y)^2 = \pi((3/10)y)^2 = (9\pi/100)y^2$.

That water has density 62.5 lb/ft^3 means $\rho g = 62.5$ (acceleration due to gravity is incorporated in the unit lb).

We have found all terms needed to fill in the integral: the work required to pump all the water out from the top of the tank is

$$W = \int_a^b \rho g D(y) A(y)\, dy = \int_0^{20} 62.5(20 - y)\left(\frac{9\pi}{100}y^2\right) dy = \frac{9\pi}{100}(62.5)\int_0^{20}(20 - y)\cdot y^2\, dy.$$

The integral is $\int_0^{20}(20y^2 - y^3)\, dy = (20y^3/3 - y^4/4)|_0^{20} = 40000/3$. Feeding this in above, $W = 75000\pi$ ft-lbs ≈ 235619 ft-lbs.

Solutions should show all of your work, not just a single final answer.

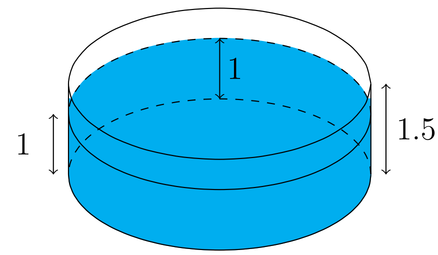
1. A cable with density 4 lb/ft is used to lift a 300 lb lump of coal up from the bottom of a mineshaft that is 1000 ft deep. Treating the coal as a concentrated point mass at the end of the cable, determine the work needed to bring it to the top of the mineshaft with the cable. (Hint: Compute the work done in lifting the cable and the coal separately.)

   (a) What is the weight of a slice of the rope with thickness Δy (or dy)?

   (b) Compute the work done in lifting the cable to the top of the mineshaft by itself.

   (c) Compute the work done in lifting the coal to the top of the mineshaft by itself, and add it to the answer in part b.

2. A circular swimming pool with diameter of 8 m. and height of 1.5 m. contains water to a height of 1 m. Compute the amount of work required to pump all the water out of the pool over the side, giving your final answer in joules[1] to the nearest integer. In metric units, the density of water is 1000 kg/m^3 and the acceleration due to gravity is $g = 9.8\,\mathrm{m/sec}^2$. Let $y = 0$ at the bottom of the tank.



(a) What is ρg?

(b) What is $D(y)$?

(c) What is $A(y)$?

(d) What should the bounds a and b be?

(e) Set up and evaluate the integral for the work required to empty the pool.

[1] Joules are the metric unit of work: it is (force)(distance) = (kg $\cdot$m/s^2)(m) = kg $\cdot$ m^2/s^2.

3. T/F (with justification): The work required to stretch a spring having spring constant k a distance x from its equilibrium (rest) position is kx.