Math 1132 Worksheet 11.8 Name: ________________________ Discussion Section: _____

11.8 Power Series

Definition. A *power series* is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ where x is a variable and the coefficients c_n are constants. More generally, for a number a, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a *power series centered at a* or a *power series in $x - a$*.

Convergence. The set of all x where a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is an interval centered at a, called the *interval of convergence*. There are three options.

(i) The series converges only at $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. (Convergence at x if $|x - a| = R$ depends on the example.)

We call R the *radius of convergence*, setting $R = 0$ in case (i) and $R = \infty$ in case (ii). Often R is found with the Ratio Test, but knowing R does **not** determine convergence at endpoints of the interval of convergence. Convergence there has to be checked separately.

Example: Determine the radius of convergence and interval of convergence for $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

Thinking about the problem:

The series converges at $x = 0$. For $x \neq 0$, we will investigate convergence at x using the Ratio Test ($a_n = (2^n/n)x^n$). After the interval of convergence is determined, convergence at its endpoints will be checked by other methods.

Doing the problem:

The problem asks for an interval of convergence of a power series. This series is $\sum_{n=1}^{\infty} a_n$ where $a_n = (2^n/n)x^n = 2^n x^n/n$, so for $x \neq 0$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)}{2^n x^n/n}\right| \\ &= \lim_{n\to\infty} 2|x|\frac{n}{n+1} \\ &= 2|x|.\end{aligned}$$

By the Ratio Test, the series converges when $2|x| < 1$ and diverges when $2|x| > 1$, so the series converges when $|x| < \frac{1}{2}$ and diverges when $|x| > \frac{1}{2}$. Therefore the radius of convergence is $\boxed{R = \frac{1}{2}}$. The inequality $2|x| < 1$ says x is in $(-1/2, 1/2)$, and we need to test the endpoints to see if the power series converges when $x = -1/2$ or $x = 1/2$. First we let $x = 1/2$. Then the power series is

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which diverges (the harmonic series). Next we let $x = -1/2$, so the power series becomes

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. Therefore the interval of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n}x^n$ is $\left[-\frac{1}{2}, \frac{1}{2}\right)$: the left endpoint is included but the right endpoint is not.

Solutions should show all of your work, not just a single final answer.

1. This question is about the power series $\sum_{n=1}^{\infty} \frac{7^{n+1}}{\sqrt{n}} x^n$.

   (a) Use the Ratio Test (for $x \neq 0$) to determine the radius of convergence of this power series.

   (b) Determine the interval of convergence of this power series, including the endpoints. Specify what convergence tests you use.

   (c) Use the work already done for $\sum_{n=1}^{\infty} \frac{7^{n+1}}{\sqrt{n}} x^n$ to determine the radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{7^{n+1}}{\sqrt{n}} x^{2n}$.

2. This question is about the power series $\sum_{n=1}^{\infty} n^2(x-2)^n$.

(a) Use the Ratio Test (for $x \neq 2$) to determine the radius of convergence of this power series.

(b) Determine the interval of convergence of this power series, including the endpoints. Specify what convergence tests you use.

3. Determine the radius of convergence and interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!}$.

4. T/F (with justification)

If $\sum_{n=0}^{\infty} c_n$ converges then $\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 1$.