11.5 Alternating Series

A series is called *alternating* if successive terms alternate in sign.

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \qquad \text{where all } b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series converges. The same test applies to $\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + \cdots$.

Note: If either condition in this test fails then the test is invalid and should not be used. For example, if the second condition fails then the series diverges since the general term of the series doesn't tend to 0, and if the first condition fails then the series may or may not converge (depends on the example).

Alternating Series Estimation Theorem. If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$ or $s = \sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series with all $b_n > 0$ such that

(i) $b_{n+1} \leq b_n$ for all n,

(ii) $\lim_{n\to\infty} b_n = 0$

then the nth remainder term $R_n = s - s_n$ has the upper bound $|R_n| \leq b_{n+1}$. In words, this says the error in approximating s by s_n is at most the size of the first omitted term.

Example: Decide if $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges or diverges.

Thinking about the problem:

The series starts off as $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ and is alternating with $b_n = \frac{1}{2n-1}$. We will check the conditions for the Alternating Series Test.

Doing the problem:

For $b_n = \frac{1}{2n-1} > 0$ we need to check $b_{n+1} \leq b_n$ for all n and $b_n \to 0$ as $n \to \infty$.

The inequality $b_{n+1} \leq b_n$ is the same as $\frac{1}{2n+1} \leq \frac{1}{2n-1}$, which is equivalent to saying $2n+1 \geq 2n-1$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \frac{1}{2x-1}$ has derivative $f'(x) = -\frac{2}{(2x-1)^2}$, which is negative for $x \geq 1$, so $f(x)$ is decreasing for $x \geq 1$.

For the limit, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$.

We can now use the Alternating Series Test to conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ converges.

Solutions should show all of your work, not just a single final answer.

1. Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^n}$ converges or diverges.

   (a) Check the series fits the conditions of the Alternating Series Test and determine what b_n is.

   (b) Using the Alternating Series Test, does the series converge or diverge?

2. Determine if the following series converge or diverge, with justification.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{\sqrt{n^2+1}}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{\sqrt{2n^3+1}}$

3. Let s_n be the nth partial sum of the series $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$, which converges by the example at the start of the worksheet.

   (a) Give a bound from above on the error $|s - s_{100}|$, as a decimal rounded to four digits after the decimal point.

   (b) Use the bound on the remainder in the alternating series estimation theorem to find an n such that $|s - s_n| \leq .01$.

4. T/F (with justification)

   The infinite series $\sum_{n=1}^{\infty} \frac{\cos n}{n}$ is alternating.

5. T/F (with justification)

   For the alternating series in problem 3, s lies between s_{100} and s_{101}.