10.1 Curves Defined by Parametric Equations

Parametric curve. A *parametric curve* is a curve in the plane described as $(x(t), y(t))$ where the x and y coordinates are both functions of a common parameter t (which could be an angle, a length, time, and so on). It is useful to describe curves parametrically because they provide flexibility in changing the direction or speed in which a curve is traced out.

Example: For the following two parametric curves

$$(1)\ x = \cos t,\ y = \sin t \text{ for } 0 \le t \le 2\pi, \quad (2)\ x = -\sin(2t),\ y = -\cos(2t) \text{ for } 0 \le t \le \frac{3\pi}{2}$$

eliminate the parameter to obtain an equation for the curve that directly relates x and y (non-parametric form of the curve) and then sketch the curve with an arrow indicating the direction it is traced out as t increases, and determine the initial and final point.

Thinking about the problem:

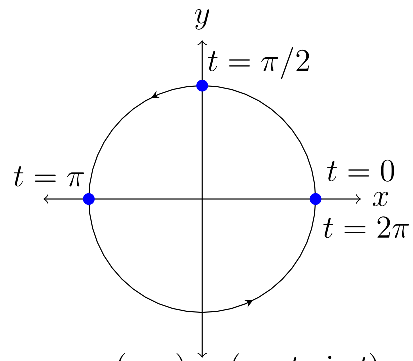
Since x and y are essentially sines and cosines (or *vice versa*) of the same value (t or $2t$), we expect the equation for the curve directly relating x and y will be a circle of radius 1. The parametric formulas for x and y will tell us the initial and final points of the traced circle, the direction the circle is traced out, and how many times the circle is traced out.

Doing the problem:

(1) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$: the curve $(x(t), y(t)) = (\cos t, \sin t)$ is part of the unit circle. In the table below we compute $(x(t), y(t))$ at $t = 0$, $t = \pi/2$, $t = \pi$, and $t = 2\pi$.

t	0	$\pi/2$	π	2π
$(\cos t, \sin t)$	$(1,0)$	$(0,1)$	$(-1,0)$	$(1,0)$

These points are marked in the figure below, which shows the curve traced out by the parametrization is a circle going counterclockwise once. The direction the curve is traced out is indicated with arrows. It starts at $(x(0), y(0)) = (1, 0)$ and ends at $(x(2\pi), y(2\pi)) = (1, 0)$.

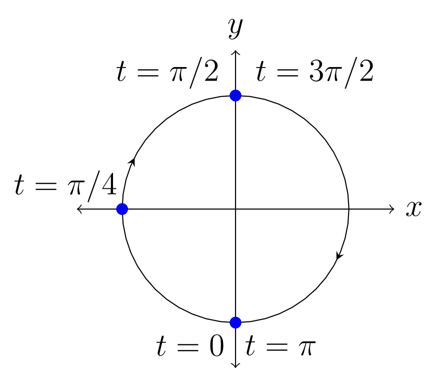


$(x, y) = (\cos t, \sin t)$

(2) $x^2+y^2 = (-\sin(2t))^2+(-\cos(2t))^2 = \sin^2(2t)+\cos^2(2t) = 1$, so the curve $(x(t), y(t)) = (-\sin(2t), -\cos(2t))$ traces out part of the unit circle. The table below shows $(x(t), y(t))$ at $t = 0$, $t = \pi/4$, $t = \pi/2$, $t = \pi$, and $t = 3\pi/2$.

t	0	$\pi/4$	$\pi/2$	π	$3\pi/2$
$(-\sin(2t), -\cos(2t))$	$(0, -1)$	$(-1, 0)$	$(0, 1)$	$(0, -1)$	$(0, 1)$

These points are marked in the figure below, which shows the curve traced out by the parametrization is a circle going clockwise *one and a half times*. The direction of increasing t is indicated with arrows. It starts at $(x(0), y(0)) = (0, -1)$ and ends at $(x(3\pi/2), y(3\pi/2)) = (0, 1)$.



$(x, y) = (-\sin(2t), -\cos(2t))$

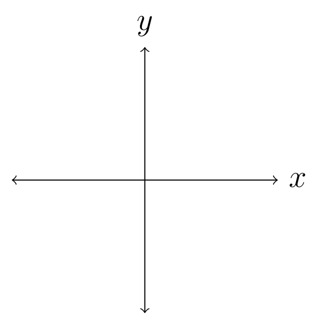
Solutions should show all of your work, not just a single final answer.

1. In the following parametric equations, eliminate the parameter to obtain a single equation only in terms of x and y.

$$x = t + 1, \quad y = 2t \text{ for } 0 \leq t \leq 1.$$

   (a) Write y in terms of x.

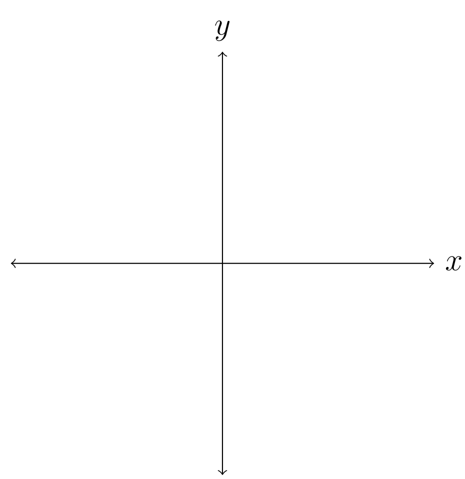
   (b) Draw a picture of this parametric curve, indicating clearly the starting and ending points and the direction of increasing values of t.



2. In the parametric equations

$$x = 3\cos t, \quad y = 3\sin t \text{ for } 0 \leq t \leq \frac{\pi}{2}.$$

eliminate the parameter to obtain a single equation in terms of x and y only. Then trace out the curve, indicating clearly the starting and ending points and the direction of increasing values of t.



3. Find a parameterization of a circle centered at the origin with radius 2 fitting the following conditions.

   (a) Oriented counterclockwise with initial point $(2, 0)$.

   (b) Oriented clockwise with initial point $(2, 0)$.

   (c) Oriented counterclockwise with initial point $(0, 2)$.

   (d) Oriented clockwise with initial point $(0, 2)$.

4. Find a parameterization of a circle centered at $(2, 1)$ with radius 3, oriented clockwise with initial point $(-1, 1)$.

5. T/F (with justification)

   As t increases, the parametric curve $(\sin t, -\cos t)$ traces out a circle counterclockwise.